

Sasakian Hypersurfaces in a Hermitian Manifold

Sushil Shukla

Department of Mathematics

Uma Nath singh Institute of engineering and technology

Veer Bahadur Singh Purvanchal University, Jaunpur 222001, India

Email: sushilcws@gmail.com

(Received October 10 , 2020)

Abstract: The object of present paper is to study the properties of Sasakian hypersurfaces in a Hermitian manifold.

Keywords: Sasakian manifold, Hypersurfaces, Hermitian manifold.

2010 AMS Classification Number: 53C15, 53C25.

1. Introduction

The almost contact metric structures are closely connected to the almost Hermitian structures. For instance, if $(N, \{\varphi, \xi, \eta, g\})$ is an almost contact metric manifold, then an almost Hermitian structure is induced on $N \times \mathbb{R}^1$. If this almost Hermitian structure is integrable, then the input almost contact metric structure is called normal. As it is known, a normal contact metric structure is called Sasakian¹. On the other hand, we can characterize the Sasakian structure by the following condition:

$$(\nabla_X \varphi)(Y) = (X, Y)\xi - \eta(Y)X, \quad X, Y \in TM.$$

As it is well known, the Sasakian structures have many remarkable properties and play a fundamental role in contact geometry. In the present paper, almost contact metric hypersurfaces in Hermitian manifolds are considered.

2. Preliminaries

Let M be an almost contact manifold equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ , a vector field ξ , a 1-form η and a compatible Riemannian metric g satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0,$$

$$g(X, Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y),$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X), \quad \text{for all } X, Y \in \chi(M).$$

An almost contact metric manifold M is a Sasakian manifold if and only if

$$(\nabla_{X\phi})(Y) = (X, Y)\xi - \eta(Y)X, \quad X, Y \in TM,$$

where ∇ is Levi-Civita connection of the Riemannian metric g .

Let us consider a differentiable manifold M of class C^∞ endowed with a tensor field F of type $(1,1)$ such that

$$\tilde{F}^2 = I, \quad \text{i.e.} \quad \tilde{F}_k^h \tilde{F}_i^k = \delta_i^h,$$

and

$$g(FX, FY) = g(X, Y).$$

Then we say that g is compatible with structure F and (F, g) is called almost Hermitian structure and the manifold M with this structure is called almost Hermitian manifold.

3. The Main Results

Theorem 3.1: *The first group of the Cartan structural equations of an almost contact metric structure on a hypersurface N in a Hermitian manifold M^{2n} is the following:*

$$dw^\alpha = w_\beta^\alpha \Lambda w^\beta + B_\gamma^{\alpha\beta} w^\gamma \Lambda w_\beta + (\sqrt{2} B_\beta^{\alpha n} + i\sigma_\beta^\alpha) w^\beta \Lambda w + (-\frac{1}{\sqrt{2}} B_n^{\alpha\beta} + i\alpha^{\alpha\beta}) w_\beta \Lambda w,$$

$$dw_\alpha = -w_\alpha^\beta \Lambda w_\beta + B_{\alpha\beta}^\gamma w_\gamma \Lambda w^\beta + (\sqrt{2} B_{\alpha n}^\beta - i\sigma_\alpha^\beta) w_\beta \Lambda w + (-\frac{1}{\sqrt{2}} B_{\alpha\beta}^n + i\alpha_{\alpha\beta}) w^\beta \Lambda w,$$

$$dw = (\sqrt{2} B_\beta^{n\alpha} - \sqrt{2} B_{n\beta}^\alpha - 2i\alpha_\beta^\alpha) w^\beta \Lambda w_\alpha + (B_{n\beta}^n + i\sigma_{n\beta}) w \Lambda w^\beta + (B_n^{n\beta} - i\alpha_n^\beta) w \Lambda w_\beta.$$

Proof: Let us use the Cartan structural equations of an almost contact metric structure on a hypersurface in an almost Hermitian manifold²:

$$dw^\alpha = w_\beta^\alpha \Lambda w^\beta + B_\gamma^{\alpha\beta} w^\gamma \Lambda w_\beta + B^{\alpha\beta\gamma} w_\beta \Lambda w_\gamma + (\sqrt{2} B_\beta^{\alpha n} + i\sigma_\beta^\alpha) w^\beta \Lambda w$$

$$+ (-\sqrt{2} \tilde{B}^{n\alpha\beta} - \frac{1}{\sqrt{2}} B_n^{\alpha\beta} - \frac{1}{\sqrt{2}} \tilde{B}^{\alpha\beta n} + i\alpha^{\alpha\beta}) w_\beta \Lambda w,$$

$$(3.1) \quad dw_\alpha = -w_\alpha^\beta \Lambda w_\beta + B_{\alpha\beta}^\gamma w_\gamma \Lambda w^\beta + B_{\alpha\beta\gamma} w^\beta \Lambda w^\gamma + (\sqrt{2} B_{\alpha n}^\beta - i\sigma_\alpha^\beta) w_\beta \Lambda w$$

$$+ (-\sqrt{2} \tilde{B}_{n\alpha\beta} - \frac{1}{\sqrt{2}} \tilde{B}_{\alpha\beta n} - \frac{1}{\sqrt{2}} B_{\alpha\beta}^n + i\alpha_{\alpha\beta}) w^\beta \Lambda w,$$

$$dw = \sqrt{2} B_{n\alpha\beta} w^\alpha \Lambda w^\beta + (\sqrt{2} B_\beta^{n\alpha} - \sqrt{2} B_{n\beta}^\alpha - 2i\alpha_\beta^\alpha) w^\beta \Lambda w_\alpha$$

$$+ (\tilde{B}_{n\beta n} + B_{n\beta}^n + i\sigma_{n\beta}) w \Lambda w^\beta + (\tilde{B}^{n\beta n} + B_n^{n\beta} - i\alpha_n^\beta) w \Lambda w_\beta,$$

here indices α, β, γ range from 1 to $n-1$.

An almost Hermitian structure is Hermitian (i.e. integrable) if and only if its Kirichenko structural tensors vanish³

$$(3.2) \quad B^{\alpha\beta\gamma} = 0, \quad B_{\alpha\beta\gamma} = 0,$$

Therefore by (3.2), (3.1) becomes

$$dw^\alpha = w_\beta^\alpha \Lambda w^\beta + B_\gamma^{\alpha\beta} w^\gamma \Lambda w_\beta + (\sqrt{2} B_\beta^{\alpha n} + i\sigma_\beta^\alpha) w^\beta \Lambda w + (-\frac{1}{\sqrt{2}} B_n^{\alpha\beta} + i\alpha^{\alpha\beta}) w_\beta \Lambda w,$$

$$dw_\alpha = -w_\alpha^\beta \Lambda w_\beta + B_{\alpha\beta}^\gamma w_\gamma \Lambda w^\beta + (\sqrt{2} B_{\alpha n}^\beta - i\sigma_\alpha^\beta) w_\beta \Lambda w + (-\frac{1}{\sqrt{2}} B_{\alpha\beta}^n + i\alpha_{\alpha\beta}) w^\beta \Lambda w,$$

$$dw = (\sqrt{2} B_\beta^{n\alpha} - \sqrt{2} B_{n\beta}^\alpha - 2i\alpha_\beta^\alpha) w^\beta \Lambda w_\alpha + (B_{n\beta}^n + i\sigma_{n\beta}) w \Lambda w^\beta + (B_n^{n\beta} - i\alpha_n^\beta) w \Lambda w_\beta.$$

Theorem 3.2: *The following are equivalent on a Sasakian hypersurface N in a Hermitian manifold:*

1. N is totally umbilical hypersurface,
2. N is totally geodesic hypersurface,
3. N is totally minimal hypersurface.

Proof: Condition for minimality of arbitrary hypersurface is⁴

$$g^{ps}\sigma_{ps}=0, p,s=1,2,\dots,2n-1.$$

And matrix of contravariant tensor on N is given by^{5,6}

$$(g^{ps}) = \begin{bmatrix} 0 & & & \\ & 0 & \dots & I_{n-1} \\ & 0 & & \\ 0 \dots 0 & 1 & 0 \dots 0 & \\ & 0 & & \\ I_{n-1} & \dots & 0 & \\ & 0 & & \end{bmatrix}.$$

Hence we have

$$\begin{aligned} g^{ps}\sigma_{ps} &= g^{\alpha\beta}\sigma_{\alpha\beta} + g^{\tilde{\alpha}\tilde{\beta}}\sigma_{\tilde{\alpha}\tilde{\beta}} + g^{\tilde{\alpha}\beta}\sigma_{\tilde{\alpha}\beta} + g^{\alpha\tilde{\beta}}\sigma_{\alpha\tilde{\beta}} + g^{nn}\sigma_{nn} \\ &= g^{\tilde{\alpha}\beta}\sigma_{\tilde{\alpha}\beta} + g^{\alpha\tilde{\beta}}\sigma_{\alpha\tilde{\beta}} + g^{nn}\sigma_{nn}, \end{aligned}$$

$$g^{ps}\sigma_{ps} = g^{nn}\sigma_{nn} = 1.\lambda = \lambda.$$

Which imply

$$g^{ps}\sigma_{ps} = 0 \Leftrightarrow \lambda = 0.$$

As the condition $\lambda = 0$ holds if and only if matrix (σ_{ps}) vanish, so this condition is a criteria for N to be a totally geodesic hypersurface in M^{2n} . By matrices (σ_{ps}) and (g^{ps}) , we have $\sigma_{ps} = \mu g_{ps, \mu-\text{const}}$ is possible if and only if $\lambda = 0$. Therefore propositions I, II and III are equivalent.

Theorem 3.3: *The almost contact metric structure on a hypersurface N in a Hermitian manifold M^{2n} is Sasakian if and only if the matrix of second fundamental form of the immersion of N into M^{2n} looks as (3.4).*

Proof: Cartan structural equation of a Sasakian structure is⁷

$$dw^\alpha = w^\alpha_\beta \Lambda w^\beta + i\delta^\alpha_\beta w \Lambda w^\beta,$$

$$dw_\alpha = -w^\beta_\alpha \Lambda w_\beta - i\delta^\beta_\alpha w \Lambda w_\beta,$$

$$dw = -2iw^\alpha \Lambda w_\alpha.$$

From these structural equations and Theorem 3.1 equations, the condition, whose fulfillment is a criterion for the structure on N to be Sasakian.

$$(3.1) \quad \sigma^\alpha_\beta = -\delta^\alpha_\beta,$$

$$(3.2) \quad \sigma^{\alpha\beta} = 0,$$

$$(3.3) \quad i\sigma^\beta_n = 0.$$

Which implies

$$\sigma^\alpha_\beta = 0 \text{ and } \sigma^\beta_n = 0.$$

Therefore besides σ_{nn} the rest of components of the second fundamental form of the immersion of N into M^{2n} vanish. Hence the matrix of the second fundamental form of a Sasakian hypersurface in a Hermitian manifold looks as

$$(3.4) \quad \sigma_{ps} = \begin{bmatrix} & 0 & \\ & 0 & \dots & 0 \\ & 0 & \\ 0 \dots 0 & \lambda & 0 \dots 0 \\ & 0 & \\ 0 & \dots & 0 \\ & 0 & \end{bmatrix}$$

Theorem 3.4: *The type number T of a Sasakian hypersurface in a Hermitian manifold is at most one.*

Proof: For a Riemannian manifold and its submanifold, the rank of the determined second fundamental form is called the type number^{8,9}. Considering the matrix of the second fundamental form of a Sasakian hypersurface in a Hermitian manifold, it is easy to see that

$$T = \text{rank}(\sigma_{ps}) \leq 1.$$

Therefore we have the theorem.

References

1. D. E. Blair, Contact Manifolds in Riemannian Geometry, *Lect. Notes Math.*, **509** (1976), 1-145.
2. A. Abu-Saleem, L. V. Stepanova and G. Banaru, Almost Contact Metric Structures on Hypersurfaces of Almost Hermitian Manifolds, *International Journal of Pure and Applied Mathematics*, **41(6)** (2007), 847-858.
3. A. Abu-Saleem and M. Banaru, Some Applications of Kirichenko Tensors, *Analele Univ. Oradea. Math.*, **17(5)** (2010), 201-208.
4. S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry I*, Inter- Science, New-York-London-Sydney, 1969.
5. A. Abu-Saleem and M. Banaru, Two Theorems on Kenmotsu Hypersurfaces in a W_3 - Manifold, *Studia Univ. Babes-Bolyai. Math.*, **51(3)** (2005), 3-11.
6. M. Banaru, Hermitian Manifolds and U-Cosymplectic Hypersurfaces Axiom, *Journal of Sichuan Normal University (Natural Science)*, **26(3)** (2003), 261-263.
7. L. V. Stepanova, *Contact Geometry of Hypersurfaces of Quasi-Kählerian Manifolds*, Mspu, Moscow, 1995.
8. H. Kurihara, The Type Number on Real Hypersurfaces in a Quaternionic Space Form, *Tsukuba Journal Math.*, **24** (2000), 127-132.
9. H. Kurihara and R. Takagi, A Note on the Type Number of Real Hypersurfaces in $Pn(C)$, *Tsukuba Journal Math.*, **22** (1998), 793-802.