

A Study of Projective and $h\nu$ -Curvature Tensors via Two Connections in Generalized Tri-recurrent Finsler Spaces

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Abstract: This paper concentrates on two theorems that illustrate the relationship between W_{jkh}^i and P_{jkh}^i in generalized trirecurrent space with Berwald and Cartan connection parameters. Moreover, the necessary and sufficient condition for W_{jkh}^i , satisfying the generalized trirecurrence property, is obtained.

Keywords: Projective curvature tensor W_{jkh}^i , $h\nu$ -curvature tensor P_{jkh}^i , Berwald's and Cartan's covariant derivatives of third order.

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1. Introduction

The curvature tensors are fundamental concepts in differential geometry, some of them include the projective curvature tensor W_{jkh}^i and Riemannian curvature tensor R_{jkh}^i . So, the relations between the various curvature tensors introduced by many authors in Finsler geometry. The recurrence and birecurrence properties for the projective curvature tensor W_{jkh}^i in Finsler space studied by Saleem and Abdallah¹⁻². The necessary condition for projective curvature tensor W_{jkh}^i that is generalized birecurrent tensor obtained by Qasem and Saleem³. Furthermore, in Berwald and Cartan of third order, Abdallah *et al.*⁴ discussed the generalized trirecurrent Finsler space for P_{jkh}^i in senses of Berwald and Cartan, respectively.

In the same vein, the conditions for some tensors to be generalized $\mathcal{B}$ –tri recurrent have been obtained by Ahmed⁵⁻⁶. The relationship between W_{jkh}^i and P_{jkh}^i in generalized $\mathcal{BP}$ –recurrent space and generalized $\mathcal{BP}$ –bi recurrent space are discussed by Abdallah⁷⁻⁸. Two theorems that clarify the relationship between P_{ijk}^h and R_{ijk}^h in generalized tri recurrent space are discussed⁹⁻¹⁰.

2. Preliminaries

Let $F_n = (M, L(x, y))$ is called an n -dimensional Finsler space if the function $L(x, y)$ satisfying the three conditions¹¹⁻¹⁶. The covariant vector y_j is given by

$$(2.1) \quad y_j = g_{kj}(x, y)y^k.$$

The metric tensor g_{ij} and its associative are connected by

$$(2.2) \quad g_{kj} g^{ki} = \delta_j^i = \begin{cases} 1 & , \quad \text{if } j = i, \\ 0 & , \quad \text{if } j \neq i. \end{cases}$$

In view of (2.1) and (2.2), we have

$$(2.3) \quad \delta_j^i y^j = y^i \quad \text{and} \quad \delta_j^i g_{ir} = g_{jr}.$$

The $(h)hv$ -torsion tensor C_{ijk} is given¹⁷

$$C_{ijk} = \frac{1}{2} \dot{\partial}_i g_{jk} = \frac{1}{4} \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k F^2$$

and satisfy

$$(2.4) \quad C_{ijk} y^i = C_{kij} y^i = C_{jki} y^i = 0.$$

Berwald's Covariant derivative $\mathcal{B}_k$ of a random ensor T_j^i w.r.t. x^k is given by

$$\mathcal{B}_k T_j^i = \partial_k T_j^i - (\dot{\partial}_r T_j^i) G_k^r + T_j^r G_{rk}^i - T_r^i G_{jk}^r,$$

where G_{rk}^i is a function called it Berwald's connection parameter. Therefore, $\mathcal{B}_k$ of the vector y^i and metric tensor g_{ij} are satisfied

$$(2.5) \quad \mathcal{B}_k y^i = 0 \quad \text{and} \quad \mathcal{B}_k g_{ij} = -2 C_{ijk|h} y^h = -2 y^h \mathcal{B}_h C_{ijk}.$$

Cartan h -covariant derivative (Cartan's second kind covariant differentiation) with respect to x^k is given¹⁸

$$X_{(k)}^i = \partial_k X^i - (\dot{\partial}_r X^i) G_k^r + X^r \Gamma_{rk}^{*i},$$

where Γ_{rk}^{*i} is a function called it Cartan's connection parameter. Therefore, h -covariant derivative of the vector y^i and metric tensor g_{ij} are vanish identically, i.e.

$$(2.6) \quad y_{(k)}^i = 0 \quad \text{and} \quad g_{ij(k)} = 0.$$

The tensor R_{jkh}^i is called (Cartan's third curvature tensor) and defined¹⁹

$$R_{jkh}^i = \partial_h \Gamma_{jk}^{*i} + (\partial_l \Gamma_{jh}^{*i}) G_k^l + G_{jm}^i (\partial_h G_k^m - G_{hl}^m G_k^l) + \Gamma_{mh}^{*i} \Gamma_{jk}^{*m} - h/k.$$

The R -Ricci tensor and curvature vector R_{jk} are connected by

$$(2.7) \quad R_{jk} y^j = R_k.$$

The tensor P_{jkh}^i is called $h\nu$ -curvature tensor which is given¹⁹

$$P_{jkh}^i = \partial_h \Gamma_{jk}^{*i} + C_{jr}^i P_{kh}^r - C_{jh|k}^i,$$

which satisfies

$$(2.8) \quad P_{jkh}^i y^j = \Gamma_{kh}^{*i} y^j = P_{kh}^i = C_{kh|r}^i y^r,$$

where P_{kh}^i is called $(\nu)h\nu$ -torsion tensor. The associate tensor P_{ijkh} of P_{jkh}^i is given by

$$(2.9) \quad P_{ijkh} = g_{ir} P_{jkh}^r.$$

The projective curvature tensor W_{jkh}^i and projective torsion tensor W_{kh}^i satisfy

$$(2.10) \quad W_{jkh}^i y^j = W_{kh}^i \quad \text{and} \quad W_{jkh}^i g_{pi} = W_{pjkh}.$$

The generalized $\mathcal{BP}$ -trirecurrent space and generalized P^h -trirecurrent space introduced by Abdallah and Hardan⁴ which characterized by

$$(2.11) \quad \begin{aligned} \mathcal{B}_l \mathcal{B}_m \mathcal{B}_n P_{jkh}^i &= a_{lmn} P_{jkh}^i + b_{lmn} (\delta_j^i g_{kh} - \delta_k^i g_{jh}) \\ &\quad - 2y^t c_{mn} \mathcal{B}_t (\delta_j^i C_{khl} - \delta_k^i C_{jhl}) - 2y^t d_{ln} \mathcal{B}_t (\delta_j^i C_{khn} - \delta_k^i C_{jhn}) \\ &\quad - 2y^t \mu_n \mathcal{B}_l \mathcal{B}_t (\delta_j^i C_{khn} - \delta_k^i C_{jhn}) \end{aligned}$$

and

$$(2.12) \quad P_{jkh(l)(m)(n)}^i = c_{lmn} P_{jkh}^i + d_{lmn} (\delta_j^i g_{kh} - \delta_k^i g_{jh}),$$

respectively, where $a_{lmn} = \mathcal{B}_l u_{mn} + u_{mn} \lambda_l$ and $b_{lmn} = \mathcal{B}_l v_{mn} + u_{mn} \mu_l$ are non-zero covariant tensors field of third order, $c_{mn} = v_{mn}$ and $d_{ln} = \mathcal{B}_l \mu_n$ are non-zero covariant tensor field of second order. $\mathcal{B}_l \mathcal{B}_m \mathcal{B}_n$ is the differential operator in sense of Berwald with respect to x^n , x^m and x^l successively. Also, $c_{lmn} = a_{lm(n)} + a_{lm} \lambda_n$ and $d_{lmn} = a_{lm} \mu_n + b_{lm(n)}$ are non-zero covariant tensors field of third order. $(l)(m)(n)$ is the differential operator in sense of Cartan with respect to x^l , x^m and x^n , successively. The above spaces are indicated by $G(\mathcal{BP}) - TRF_n$ and $G(P^h) - TRF_n$. Also, these tensors which satisfy the conditions (2.11) and (2.12) are called a generalized $\mathcal{B}$ -trirecurrent tensor and generalized h -trirecurrent tensor, respectively.

Transvecting the condition (2.11) by y^j , (2.1), (2.3), (2.4), (2.5) and (2.8), we get

$$(2.13) \quad \mathcal{B}_l \mathcal{B}_m \mathcal{B}_n P_{kh}^i = a_{lmn} P_{kh}^i + b_{lmn} (y^i g_{kh} - \delta_k^i y_h) - 2y^t c_{mn} \mathcal{B}_t (y^i C_{khl}) \\ - 2y^t d_{ln} \mathcal{B}_t (y^i C_{khm}) - 2y^t \mu_n \mathcal{B}_l \mathcal{B}_t (y^i C_{khn}).$$

Transvecting (2.12) by g_{ir} , using (2.2), (2.3), (2.6) and (2.9), we get

$$(2.14) \quad P_{ijkh(l)(m)(n)} = c_{lmn} P_{ijkh} + d_{lmn} (g_{rj} g_{kh} - g_{rk} g_{jh}).$$

Transvecting (2.12) by y^j , using (2.1), (2.3), (2.5) and (2.8), we get

$$(2.15) \quad P_{kh(l)(m)(n)}^i = c_{lmn} P_{kh}^i + d_{lmn} (y^i g_{kh} - \delta_k^i y_h).$$

This paper studies the relationship between W_{jkh}^i and P_{jkh}^i in two Finsler spaces based on previous formulas, i.e. study the relationship between them in generalized Finsler spaces of third order in Berwald and Cartan senses.

3. Main Results

This paper discusses the condition for projective curvature tensor W_{jkh}^i which is generalized trirecurrent tensor with two connections. For a Riemannian space V_4 , hv -curvature tensor P_{jkh}^i and projective curvature tensor W_{jkh}^i can be expressed as follows²⁰

$$(3.1) \quad P_{jkh}^i = W_{jkh}^i - \frac{1}{3} (\delta_h^i R_{jk} - \delta_k^i R_{jh}).$$

Transvecting (3.1) by g_{ir} , using (2.3), (2.9) and (2.10), we get

$$(3.2) \quad P_{rjkh} = W_{rjkh} - \frac{1}{3} (R_{jk} g_{rh} - R_{jh} g_{rk}).$$

Transvecting (3.1) by y^j , using (2.7), (2.8) and (2.10), we get

$$(3.3) \quad P_{kh}^i = W_{kh}^i - \frac{1}{3} (\delta_h^i R_k - \delta_k^i R_h).$$

In next theorem we obtain the condition for W_{jkh}^i that satisfies the generalized trirecurrence property in Berwald sense.

Theorem 3.1. *In $G(\mathcal{BP}) - TRF_n$, the projective curvature tensor W_{jkh}^i is a generalized $\mathcal{B}$ -tri recurrent if and only if the tensor $(\delta_h^i R_k - \delta_k^i R_h)$ behaves as tri recurrent.*

Proof. Let us consider a $G(\mathcal{BP}) - TRF_n$. Taking $\mathcal{B}$ -covariant derivative for (3.1) thrice with respect to x^n , x^m and x^l successively, then applying (2.11), we get

$$\mathcal{B}_l \mathcal{B}_m \mathcal{B}_n W_{jkh}^i - \frac{1}{3} \mathcal{B}_l \mathcal{B}_m \mathcal{B}_n (\delta_h^i R_{jk} - \delta_k^i R_{jh}) = a_{lmn} P_{jkh}^i$$

$$+b_{lmn}(\delta_j^i g_{kh} - \delta_k^i g_{jh}) - 2y^t c_{mn} \mathcal{B}_t(\delta_j^i C_{khl} - \delta_k^i C_{jhl}) \\ - 2y^t d_{ln} \mathcal{B}_t(\delta_j^i C_{khm} - \delta_k^i C_{jhm}) - 2y^t \mu_n \mathcal{B}_l \mathcal{B}_t(\delta_j^i C_{khm} - \delta_k^i C_{jhm}).$$

Using (3.1) in above equation, we get

$$\mathcal{B}_l \mathcal{B}_m \mathcal{B}_n W_{jkh}^i = a_{lmn} W_{jkh}^i + b_{lmn}(\delta_j^i g_{kh} - \delta_k^i g_{jh}) \\ - 2y^t c_{mn} \mathcal{B}_t(\delta_j^i C_{khl} - \delta_k^i C_{jhl}) - 2y^t d_{ln} \mathcal{B}_t(\delta_j^i C_{khm} - \delta_k^i C_{jhm}) \\ - 2y^t \mu_n \mathcal{B}_l \mathcal{B}_t(\delta_j^i C_{khm} - \delta_k^i C_{jhm})$$

if and only if

$$(3.4) \quad \mathcal{B}_l \mathcal{B}_m \mathcal{B}_n (\delta_h^i R_{jk} - \delta_k^i R_{jh}) = a_{lmn} (\delta_h^i R_{jk} - \delta_k^i R_{jh}).$$

Above equation refers to the behavior of $(\delta_h^i R_{jk} - \delta_k^i R_{jh})$ is trirecurrent.

Now, we have a corollary related to previous theorem. Taking $\mathcal{B}$ -covariant derivative for (3.3) thrice with respect to x^n, x^m and x^l successively, then using (2.13), we obtain

$$\mathcal{B}_l \mathcal{B}_m \mathcal{B}_n W_{kh}^i - \frac{1}{3} \mathcal{B}_l \mathcal{B}_m \mathcal{B}_n (\delta_h^i R_{jk} - \delta_k^i R_{jh}) = a_{lmn} P_{kh}^i + b_{lmn} (y^i g_{kh} - \delta_k^i y_h) \\ - 2y^t c_{mn} \mathcal{B}_t (y^i C_{khl}) - 2y^t d_{ln} \mathcal{B}_t (y^i C_{khm}) - 2y^t \mu_n \mathcal{B}_l \mathcal{B}_t (y^i C_{khm}).$$

Using (3.3) in above equation, we get

$$(3.5) \quad \mathcal{B}_l \mathcal{B}_m \mathcal{B}_n W_{kh}^i = a_{lmn} W_{kh}^i + b_{lmn} (y^i g_{kh} - \delta_k^i y_h) \\ - 2y^t c_{mn} \mathcal{B}_t (y^i C_{khl}) - 2y^t d_{ln} \mathcal{B}_t (y^i C_{khm}) - 2y^t \mu_n \mathcal{B}_l \mathcal{B}_t (y^i C_{khm})$$

if and only if

$$(3.6) \quad \mathcal{B}_l \mathcal{B}_m \mathcal{B}_n (\delta_h^i R_k - \delta_k^i R_h) = a_{lmn} (\delta_h^i R_k - \delta_k^i R_h).$$

The equation (3.6) refers to the behavior of $(\delta_h^i R_k - \delta_k^i R_h)$ is trirecurrent. Thus, we conclude

Corollary 3.1. In $G(\mathcal{BP}) - TRF_n$, Berwald's covariant derivative of third order for the torsion tensor W_{kh}^i is given by (3.5) if and only if the tensor $(\delta_h^i R_k - \delta_k^i R_h)$ behaves as trirecurrent.

In next theorem we obtain the condition for W_{jkh}^i that be generalized trirecurrent in Cartan sense.

Theorem 3.2. In $G(P^h) - TRF_n$, the projective curvature tensor W_{jkh}^i is a generalized h -tri-recurrent if and only if the tensor $(\delta_h^i R_{jk} - \delta_k^i R_{jh})$ behaves as trirecurrent.

Proof. Let us consider a $G(P^h) - TRF_n$. Taking h -covariant derivative for (3.1) thrice with respect to x^l, x^m and x^n , successively, then applying (2.12), we get

$$W_{jkh(l)(m)(n)}^i - \frac{1}{3} (\delta_h^i R_{jk} - \delta_k^i R_{jh})_{(l)(m)(n)} = c_{lmn} P_{jkh}^i + d_{lmn} (\delta_j^i g_{kh} - \delta_k^i g_{jh}).$$

Using (3.1) in above equation, we get

$$W_{jkh(l)(m)(n)}^i = c_{lmn} W_{jkh}^i + d_{lmn} (\delta_j^i g_{kh} - \delta_k^i g_{jh}),$$

if and only if

$$(3.7) \quad (\delta_h^i R_{jk} - \delta_k^i R_{jh})_{(l)(m)(n)} = c_{lmn} (\delta_h^i R_{jk} - \delta_k^i R_{jh}).$$

The last equation refers to the behavior of $(\delta_h^i R_{jk} - \delta_k^i R_{jh})$ is trirecurrent.

Now, we have a corollary related to previous theorem. Taking h -covariant derivative for (3.2) thrice with respect to x^l, x^m and x^n , successively, then using (2.14), we get

$$\begin{aligned} W_{rjkh(l)(m)(n)} - \frac{1}{3} (R_{jk} g_{rh} - R_{jh} g_{rk})_{(l)(m)(n)} \\ = c_{lmn} P_{rjkh} + d_{lmn} (g_{rj} g_{kh} - g_{rk} g_{jh}). \end{aligned}$$

Using (3.2) in above equation, we get

$$(3.8) \quad W_{rjkh(l)(m)(n)} = c_{lmn} W_{rjkh} + d_{lmn} (g_{rj} g_{kh} - g_{rk} g_{jh}),$$

if and only if

$$(3.9) \quad (R_{jk} g_{rh} - R_{jh} g_{rk})_{(l)(m)(n)} = c_{lmn} (R_{jk} g_{rh} - R_{jh} g_{rk}).$$

Taking h -covariant derivative for (3.3) thrice with respect to x^l, x^m and x^n , successively, then using (2.15), we obtain

$$W_{kh(l)(m)(n)}^i - \frac{1}{3} (\delta_h^i R_k - \delta_k^i R_h)_{(l)(m)(n)} = c_{lmn} P_{kh}^i + d_{lmn} (y^i g_{kh} - \delta_k^i y_h).$$

Using (3.3) in above equation, we get

$$(3.10) \quad W_{kh(l)(m)(n)}^i = c_{lmn} W_{kh}^i + d_{lmn} (y^i g_{kh} - \delta_k^i y_h)$$

if and only if

$$(3.11) \quad (\delta_h^i R_k - \delta_k^i R_h)_{(l)(m)(n)} = c_{lmn} (\delta_h^i R_k - \delta_k^i R_h).$$

The equations (3.9) and (3.11) refer to the behavior of $(R_{jk} g_{rh} - R_{jh} g_{rk})$ and $(\delta_h^i R_k - \delta_k^i R_h)$ are trirecurrent. Thus, we conclude

Corollary 3.2. *In $G(P^h) - TRF_n$, Cartan's covariant derivative of third order for the associate curvature tensor W_{rjkh} and torsion tensor W_{kh}^i are given by (3.8) and (3.10) if and only if the tensors $(R_{jk} g_{rh} - R_{jh} g_{rk})$ and $(\delta_h^i R_k - \delta_k^i R_h)$ behave as trirecurrent, respectively.*

4. Conclusion

We proved that W_{jkh}^i satisfies the generalized trirecurrence property in $G(BP) - TRF_n$ and $G(P^h) - TRF_n$. Further, we obtained Berwald's and Cartan's covariant derivatives of third order for the associate curvature tensor

W_{rjkh} and torsion tensor W_{kh}^i are non - vanishing. This work will allow researchers to study these relationships between various curvature tensors in generalized Finsler spaces of higher orders in two connections.

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