

Boundedness and Uniqueness of Lagrange Multipliers for Multiobjective Optimization Problems with Vanishing Constraints

Abeka Khare

Department of Mathematics,
Late Rajiv Gandhi Govt. College, Kalapipal, Dist. Shajapur, India
Email: abekakhare2012@gmail.com

Triloki Nath*

Department of Mathematics and Statistics,
Deen Dayal Upadhyaya Gorakhpur University, Gorakhpur, India
*Corresponding Author E-mail: tnverma07@gmail.com

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Abstract: In this paper, we study a multiobjective optimization problem with vanishing constraints. Here, we introduce proper M- and S-regularity conditions, which provide the necessary and sufficient conditions for nonemptiness and boundedness of some sets of Lagrange multipliers. Further, we investigate the uniqueness of the Lagrange multipliers by introducing the concept of strict-Mangasarian Fromovitz constraint qualification for the multiobjective optimization problem with vanishing constraints. We show that this strict constraint qualification is equivalent to the uniqueness of the Lagrange multipliers.

Keywords: Multiobjective Optimization Problems, Vanishing constraints, Optimality conditions, Constraint qualification.

1. Introduction

In this paper, we deal with the multiobjective optimization problem with vanishing constraints (or MOPVC in short), having the following mathematical form:

$$\begin{aligned}
 (1) \quad & \min f(x) = (f_1(x), \dots, f_p(x)), \\
 \text{such that} \quad & g_i(x) \leq 0 \quad ; \quad \forall i = 1, 2, \dots, l \\
 & h_j(x) = 0 \quad ; \quad \forall j = 1, 2, \dots, m \\
 & H_i(x) \geq 0 \quad ; \quad \forall i = 1, 2, \dots, k \\
 & G_i(x)H_i(x) \leq 0 \quad ; \quad \forall i = 1, 2, \dots, k
 \end{aligned}$$

where $f: \mathbb{R}^n \rightarrow \mathbb{R}^p$, $g_i: \mathbb{R}^n \rightarrow \mathbb{R}$, $h_i: \mathbb{R}^n \rightarrow \mathbb{R}$, $G_i, H_i: \mathbb{R}^n \rightarrow \mathbb{R}$ are continuously differentiable functions.

The main focus of this paper is twofold: first is to search for restrictions for an MOPVC to have bounded Lagrange multipliers, and second is to investigate the conditions under which an MOPVC has unique Lagrange multipliers.

Although some notable literature is available about the boundedness and uniqueness of Lagrange multipliers for standard nonlinear programs, there is a shortage of such work about MOPVC, even for the scalar mathematical program with vanishing constraints (MPVC). Gauvin¹ investigated for the first time that a standard nonlinear program having equality and inequality constraints has a nonempty and bounded Lagrange multiplier set, provided that the well-known Mangasarian-Fromovitz constraint qualification holds at the concerned point. Zowe and Kurcyusz² showed that the set of Lagrange multipliers for a set-constrained nonlinear program is nonempty and bounded under a regularity assumption. This proposed regularity condition is preserved under small perturbations. Later, Nguyen *et al.*³ proved the same result for a nonlinear program having a locally Lipschitz objective and inequality constraints with continuously differentiable equality constraints. Further, Pappalardo⁴ obtained bounds for Lagrange multipliers for a nonsmooth Lipschitz program. He showed that the MFCQ (modified using Clarke subdifferential) is equivalent to the nonemptiness and boundedness of the multiplier set. Luskan *et al.*⁵ stated the necessity of the boundedness of the Lagrange multipliers and elaborated that it is an important tool in the interior point method. Dutta and Lalitha⁶ studied the boundedness of Lagrange multiplier rules for vector optimization. Here, we have used a similar approach⁶ to prove the boundedness result for MOPVC. Anitescu⁷ studied a nonlinear program having an unbounded Lagrange multiplier set. In the paper⁷, the author converted the problem into a penalized problem by adding a penalty term to the objective function and showed that this new problem has bounded Lagrange multipliers. This whole investigation is carried out under the quadratic growth assumptions. Recently, Movahedian⁸ studied a mathematical program with a set constraint and provided several constraint qualifications to investigate the nonemptiness and boundedness of the multiplier set together with their exact upper bound. They also applied their results to an MPEC problem.

In this paper, we also emphasize the uniqueness of the Lagrange multiplier, which may play an important role in further analysis. The proposed work is motivated by Bigi and Castellani⁹, who worked on a multiobjective nonlinear program having equality and inequality constraints

and proposed a regularity condition called the strictly Mangasarian-Fromovitz regularity condition (SMFRC) for this multiobjective program and proved the uniqueness of the Lagrange multiplier. Although the notion of SMFRC has been used by Kyparisis¹⁰, where it was introduced as strictly Mangasarian-Fromovitz constraint qualification (SMFCQ) for the scalar case and used to establish the uniqueness of Lagrange multipliers for a single objective nonlinear program, but in the paper⁹, it was generalized for the multiobjective program. Also, the paper⁹ motivated us to study the MOPVC and to introduce MOPVC-SMFCQ for the study of the uniqueness of Lagrange multipliers.

This paper is organized as follows. Section 2 contains the preliminary results, which are required to establish our main results. Sections 3 and 4 are devoted to establishing the boundedness and uniqueness results for Lagrange multipliers, respectively.

2. Preliminary

Here, we recall basic terms and results, which are required to understand the work of this paper. Let us denote by $\mathcal{X}$, the feasible set of problem (1), given as:

$$(2) \quad \mathcal{X} = \left\{ x \in \mathbb{R}^n \left| \begin{array}{ll} g_i(x) & \leq 0 \quad ; \quad \forall i = 1, 2, \dots, l \\ h_j(x) & = 0 \quad ; \quad \forall j = 1, 2, \dots, m \\ H_i(x) & \geq 0 \quad ; \quad \forall i = 1, 2, \dots, k \\ H_i(x)G_i(x) & \leq 0 \quad ; \quad \forall i = 1, 2, \dots, k \end{array} \right. \right\}$$

Now, for MOPVC (1), the efficient solution is defined as follows -

Definition 2.1. Let $x^* \in \mathcal{X}$ be a feasible solution of the MOPVC (1). Then x^* is said to be an efficient solution of (1) if and only if there is no $x \in \mathcal{X}$ satisfying

$$\begin{aligned} & f_i(x) \leq f_i(x^*), \quad \forall i = 1, \dots, p \\ & \text{and} \quad f_i(x) < f_i(x^*), \quad \text{for at least one } i. \end{aligned}$$

Next, suppose $x^* \in \mathcal{X}$ be a feasible point. Then we need the various index sets defined for MPVC, as follows:

$$\begin{aligned} I_g(x^*) &:= \{i \in (1, \dots, l) \mid g_i(x^*) = 0\}, \\ I_H(x^*) &:= \{i \in (1, \dots, k) \mid H_i(x^*) = 0\}, \\ I_G(x^*) &:= \{i \in (1, \dots, k) \mid G_i(x^*) = 0\}, \\ I_{+0}(x^*) &:= \{i \in (1, \dots, k) \mid H_i(x^*) > 0, G_i(x^*) = 0\} \\ I_{+-}(x^*) &:= \{i \in (1, \dots, k) \mid H_i(x^*) > 0, G_i(x^*) < 0\} \\ I_{0+}(x^*) &:= \{i \in (1, \dots, k) \mid H_i(x^*) = 0, G_i(x^*) > 0\} \end{aligned}$$

$$I_{00}(x^*) := \{i \in (1, \dots, k) \mid H_i(x^*) = 0, G_i(x^*) = 0\}$$

$$I_{0-}(x^*) := \{i \in (1, \dots, k) \mid H_i(x^*) = 0, G_i(x^*) < 0\}.$$

Now, we have a KKT-type necessary optimality condition for MOPVC (1), from¹¹, which holds under the standard generalized Guignard constraint qualification (GGCQ). It will be used to show the uniqueness of Lagrange multipliers.

Theorem 2.1. *Let $x^* \in X$ be an efficient solution of the MOPVC (1). If the standard GGCQ holds at x^* . Then there exist Lagrange multipliers $\alpha_i \in \mathbb{R}, (i \in P), \lambda_i \in \mathbb{R}, (i \in L), \mu_i \in \mathbb{R}, (i \in M), \eta_i^G, \eta_i^H \in \mathbb{R}, (i \in N)$, such that*

$$(3) \quad \sum_{i=1}^p \alpha_i \nabla f_i(x^*) + \sum_{i=1}^l \lambda_i \nabla g_i(x^*) + \sum_{i=1}^m \mu_i \nabla h_i(x^*) + \sum_{i=1}^k \eta_i^G \nabla G_i(x^*) - \sum_{i=1}^k \eta_i^H \nabla H_i(x^*) = 0$$

where

$$(4) \quad \begin{aligned} \alpha_i &> 0 && ; && i \in \mathcal{P}, \\ g_i(x^*) &\leq 0, \lambda_i \geq 0 && ; && \lambda_i g_i(x^*) = 0; i \in \mathcal{L}, \\ h_i(x^*) &= 0 && ; && i \in \mathcal{M}, \\ \eta_i^H(x^*) &= 0 && ; && i \in I_+(x^*) \\ \eta_i^H &\text{ is free} && ; && i \in I_{0+}(x^*), \\ \eta_i^H(x^*) &\geq 0 && ; && i \in I_{00}(x^*) \cup I_{0-}(x^*) \\ \eta_i^H H_i(x^*) &= 0 && ; && i \in \mathcal{N}, \\ \eta_i^G(x^*) &= 0 && ; && i \in I_0(x^*) \cup I_{+-}(x^*), \\ \eta_i^G(x^*) &\geq 0 && ; && i \in I_{+0}(x^*), \\ \eta_i^G G_i(x^*) &= 0 && ; && i \in \mathcal{N}. \end{aligned}$$

Also,

$$\mathcal{P} = \{1, \dots, p\}$$

$$\mathcal{L} = \{1, \dots, l\}$$

$$\mathcal{M} = \{1, \dots, m\}$$

$$\mathcal{N} = \{1, \dots, k\}$$

The multipliers $(\alpha, \lambda, \mu, \eta^G, \eta^H)$ satisfying (3) and (4) are said to be strong KKT-type multipliers for MOPVC, or simply KKT multipliers. We consider the following sets of multipliers.

$$E(x^*) = \{(\alpha, \lambda, \mu, \eta^G, \eta^H) \mid (3), (4) \text{ holds} \},$$

$$E_r(x^*) = \{(\alpha, \lambda, \mu, \eta^G, \eta^H) \mid \alpha_r = 1; r \in \mathcal{P}\}.$$

Here, $E(x^*)$ denotes the set of all KKT multipliers associated with x^* . It is easy to show that this set is unbounded, because if we choose $(\alpha, \lambda, \mu, \eta^G, \eta^H) \in E(x^*)$, then $(t\alpha, t\lambda, t\mu, t\eta^G, t\eta^H) \in E(x^*)$ for any real

number $t > 0$. The unboundedness of multipliers is not a desired property, at least from an algorithmic point of view. In $E(x^*)$ if we assign $\alpha_r = 1$ for some $r \in \mathcal{P}$, then this set $E(x^*)$ can be made bounded. We denote this bounded set as $E_r(x^*)$.

3. Boundedness of Lagrange multipliers for MOPVC

Now analogous to Basic regularity condition (BRC)⁶, we define MOPVC - proper M-regularity condition (MOPVC-PMRC) and MOPVC-proper S-regularity condition (MOPVC-PSRC) for a multiobjective optimization problem with vanishing constraints (1). The motivation behind these definitions is from¹².

Definition 3.1. Proper M-regularity condition: Let x^* be a feasible point of MOPVC, then we say that the MOPVC-PMRC holds at x^* , if there exists $r \in \mathcal{P}$ such that the only multipliers that satisfy

1. $\sum_{i \in \mathcal{P}, i \neq r} \alpha_i \nabla f_i(x^*) + \sum_{i=1}^l \lambda_i \nabla g_i(x^*) + \sum_{i=1}^m \mu_i \nabla h_i(x^*) - \sum_{i=1}^k \eta_i^H \nabla H_i(x^*) + \sum_{i=1}^k \eta_i^G \nabla G_i(x^*) = 0$.
2. $\alpha_i \geq 0$ ($i \in \mathcal{P}, i \neq r$), $\lambda_i \geq 0$ ($i \in I_g(x^*)$), $\lambda_i = 0$ ($i \notin I_g(x^*)$), $\eta_i^G = 0$ ($i \in I_{+-}(x^*) \cup I_{0-}(x^*)$), $\eta_i^G \geq 0$ ($i \in I_{+0}(x^*)$), $\eta_i^H = 0$ ($i \in I_+(x^*)$), $\eta_i^H \geq 0$ ($i \in I_{0-}(x^*) \cup I_{00}(x^*)$), η_i^H is free ($i \in I_{0+}(x^*)$), $\eta_i^G \eta_i^H = 0$ ($i \in I_{00}(x^*)$), $\eta_i^G = 0 = \eta_i^H$ for all $i \in \mathcal{K}$, $\alpha_i = 0$ for all $i \in \mathcal{P}, i \neq r$, $\lambda_i = 0$ for all $i \in \mathcal{L}$, $\mu_i \in \mathbb{R}$ ($i \in \mathcal{M}$), $\mu_i = 0$ for all $i \in \mathcal{M}$.

Multipliers $(\alpha, \lambda, \mu, \eta^H, \eta^G)$ for which the above proper M-regularity condition holds, are said to be Proper M-multipliers.

Definition 3.2. Proper S-regularity condition: Let x^* be a feasible point for MOPVC (1). Then we say that MOPVC-PSRC holds at x^* , if the conditions (1) and (2) of Definition 3.1 hold with $\eta_i^G = 0$, $\eta_i^H \geq 0$ ($i \in I_{00}(x^*)$) and multipliers $(\alpha, \lambda, \mu, \eta^H, \eta^G)$ for which above proper S-regularity condition is satisfied, are said to be proper S-multipliers.

The above properties are called MOPVC-proper regularity conditions (MOPVC-PRCs) with respect to the index r . If any of the MOPVC-PRCs hold at x^* , then it is easy to see that $\{\nabla h_i(x^*)\}$ are linearly independent, for all $i \in M$. Further, if either of the MOPVC-PRCs holds at any feasible point x^* , then $p < k$ (by using the argument similar to that given⁶).

Now, here we have the main result of this section, which guarantees the non-emptiness and boundedness of the Lagrange multiplier set for MOPVC under any one of MOPVC-PRCs.

Theorem 3.1. *Let x^* be an efficient solution for MOPVC with smooth data. Further, assume that MOPVC-PMRC (or MOPVC-PSRC) holds at x^* . Then there exists a set of proper Multipliers (or proper S -multipliers), which is nonempty and bounded.*

Proof. To prove this theorem, we adopt the approach given⁶, where the result has been proved for a smooth nonlinear vector optimization problem. Suppose at a point x^* , MOPVC-PMRC holds, then there exists a scalar $r \in \mathcal{P}$ such that

$$\begin{aligned} \alpha_i \geq 0 \ (i \in \mathcal{P}, i \neq r), \lambda_i \geq 0 \ (i \in I_g(x^*)), \lambda_i = 0 \ (i \notin I_g(x^*)), \mu_i \in \mathbb{R} \ (i \in \mathcal{M}), \\ \eta_i^G = 0 \ (i \in I_{+-}(x^*) \cup I_{0-}(x^*)), \eta_i^G \geq 0 \ (i \in I_{+0}(x^*)), \\ \eta_i^H = 0 \ (i \in I_+(x^*)), \eta_i^H \geq 0 \ (i \in I_{0-}(x^*) \cup I_{00}(x^*)), \eta_i^H \text{ is free} \\ (i \in I_{0+}(x^*)) \text{ and} \end{aligned}$$

$$\begin{aligned} \sum_{i \in \mathcal{P}, i \neq r} \alpha_i \nabla f_i(x^*) + \sum_{i=1}^l \lambda_i \nabla g_i(x^*) + \sum_{i=1}^m \mu_i \nabla h_i(x^*) \\ - \sum_{i=1}^k \eta_i^H \nabla H_i(x^*) + \sum_{i=1}^k \eta_i^G \nabla G_i(x^*) = 0 \end{aligned}$$

implies that $\alpha_i = 0, \lambda_i = 0, \mu_i = 0, \eta_i^G = 0 = \eta_i^H$ for all i .

On the other hand, if x^* is an efficient solution for MOPVC (1), Then x^* will be optimal solution to the following scalar problem¹³.

$$\begin{aligned} \min f_r(x) \\ \text{s.t. } f_i(x) \leq f_i(x^*) ; i \in \mathcal{P}, i \neq r, \\ g_i(x) \leq 0 ; \forall i \in \mathcal{L}, \\ h_i(x) = 0 ; \forall i \in \mathcal{M}, \\ H_i(x) \geq 0 ; \forall i \in \mathcal{K}, \\ G_i(x)H_i(x) \leq 0 ; \forall i \in \mathcal{K}. \end{aligned}$$

So, by applying Fritz John's optimality condition¹² to the above problem, we have scalars $\alpha_r \geq 0, \alpha_i \geq 0 \ (i \in \mathcal{P}, i \neq r), \lambda_i \geq 0 \ \forall i \in I_g(x^*), \lambda_i = 0 \ \forall i \notin I_g(x^*)$

and

$$\begin{aligned} \eta_i^G = 0 \ (\forall i \in I_{+-}(x^*) \cup I_{0-}(x^*) \cup I_{0+}(x^*)), \eta_i^G \geq 0 \ (\forall i \in I_{+0}(x^*) \cup \\ I_{00}(x^*)), \eta_i^H = 0 \ (\forall i \in I_+(x^*)), \eta_i^H \geq 0 \ (\forall i \in I_{0-}(x^*)), \eta_i^H \text{ is free} \\ (\forall i \in I_{0+}(x^*)), \eta_i^H \eta_i^G = 0 \ (\forall i \in I_{00}(x^*)) \end{aligned}$$

$$\begin{aligned} (5) \quad \alpha_r \nabla f_r(x^*) + \sum_{i \in \mathcal{P}, i \neq r} \alpha_i \nabla f_i(x^*) + \sum_{i \in \mathcal{L}} \lambda_i \nabla g_i(x^*) \\ + \sum_{i \in \mathcal{M}} \mu_i \nabla h_i(x^*) + \sum_{i \in \mathcal{K}} \eta_i^G \nabla G_i(x^*) - \sum_{i \in \mathcal{K}} \eta_i^H \nabla H_i(x^*) = 0. \end{aligned}$$

and

$$\lambda_i g_i(x^*) = 0, \forall i \in \mathcal{L}$$

Because of the MOPVC-PMRC property at x^* , we have $\alpha_r > 0$. Without loss of generality, we may assume that $\alpha_r = 1$. Hence, we have a set of multipliers of the form

$$(\alpha_1, \dots, \alpha_{r-1}, 1, \alpha_{r+1}, \lambda_1, \dots, \lambda_l, \mu_1, \dots, \mu_m, \eta_1^G, \dots, \eta_k^G, \eta_1^H, \dots, \eta_k^H)$$

that is $E_r(x^*)$ is non-empty. Now, it remains to show that $E_r(x^*)$ is bounded. Assume, on the contrary, that $E_r(x^*)$ is unbounded. We can write (5) in a compact form as

$$(6) \quad \nabla f_r(x^*) + JF_r(x^*)^T v = 0,$$

where

$$F_r(x^*) = (f_1(x^*), \dots, f_{r-1}(x^*), f_{r+1}(x^*), \dots, f_p(x^*), \\ g_1(x^*), \dots, g_l(x^*), h_1(x^*), \dots, h_m(x^*), G_1(x^*), \dots, G_k(x^*), H_1(x^*), \dots, H_k(x^*))$$

and $JF_r(x^*)$ is the Jacobian matrix of F at x^* with T denoting the transpose of the matrix. The vector v is

$$(\alpha_1, \dots, \alpha_{r-1}, \alpha_{r+1}, \lambda_1, \dots, \lambda_l, \mu_1, \dots, \mu_m, \eta_1^G, \dots, \eta_k^G, \eta_1^H, \dots, \eta_k^H)$$

Since the set of multipliers $E_r(x^*)$ is unbounded; therefore, we can find a sequence $\{v_s\}$ as

$$v_s = (\alpha_1^s, \dots, \alpha_{r-1}^s, \alpha_{r+1}^s, \lambda_1^s, \dots, \lambda_l^s, \mu_1^s, \dots, \mu_m^s, \eta_1^{Gs}, \dots, \eta_k^{Gs}, \eta_1^{Hs}, \dots, \eta_k^{Hs})$$

such that $\|v_s\| \rightarrow +\infty$. Now, we choose the sequence $\{w_s\}$ with $w_s = \frac{v_s}{\|v_s\|}$, so from (6), we have

$$0 = \frac{1}{\|v_s\|} \nabla f_r(x^*) + JF_r(x^*)^T w_s$$

Since $\{w_s\}$ is a bounded sequence, it has a convergent subsequence converging to $w_0 \neq 0$. Therefore, in the limiting case, we have

$$0 = JF_r(x^*)^T w_0$$

Since w_0 is nonzero, therefore it contradicts the MOPVC-PMRC assumption at x^* with respect to r . Hence $E_r(x^*)$ is bounded.

Similarly, for the MOPVC-PSRC case, by using¹², we can show the desired result. ■

We illustrate the above result in the following example.

Example 3.1. Consider the following MOPVC.

$$\begin{aligned} & \min(f_1, f_2) \\ & \text{where } f_1(x) = x_1^2 + x_2 + x_4^2 \\ & \quad f_2(x) = x_4 \\ & \quad g(x) = -x_1 \leq 0 \\ & \quad h(x) = x_1^2 + 2x_2 = 0 \end{aligned}$$

$$H(x) = x_2^2 + x_5 \geq 0$$

$$\text{and } G(x)H(x) = -x_3(x_2^2 + x_5) \leq 0.$$

Clearly, $x^* = (0,0,0,0,0)$ is an efficient solution. If we consider the index $r = 1$, then

$$\begin{aligned} \alpha_2 \nabla f_2(x^*) + \lambda \nabla g(x^*) + \mu \nabla h(x^*) + \eta^G \nabla G(x^*) - \eta^H \nabla H(x^*) &= 0 \\ \alpha_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \eta^G \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} - \eta^H \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\ \Rightarrow \alpha_2 = \lambda = \mu = \eta^G = \eta^H &= 0, \end{aligned}$$

That is, MOPVC-PMRC holds at x^* . Now, we try to find the set of proper M-multipliers,

$$E_1(0,0,0,0,0) = \{(1, \alpha_2, \lambda, \mu, \eta^G, \eta^H)\}$$

such that

$$\begin{aligned} 1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \eta^G \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} - \eta^H \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \\ E_1(0,0,0,0,0) &= \{(1,0,0,-1/2,0,0)\} \end{aligned}$$

This clearly shows that the set of Lagrange multipliers $E_1(0,0,0,0,0)$ is bounded. ■

Next, it is natural to ask about the converse of Theorem 3.1. We investigate it positively. To prove this result, we use the approach¹⁴, where the result has been derived for non-smooth optimization problems using Michel-Penot subdifferentials of the involved functions. Here, all the involved functions are continuously differentiable, so we are doing this for MOPVC using the gradient of the functions.

Theorem 3.2. *Let x^* be a feasible point for smooth MOPVC. If there exists a set of proper multipliers that is nonempty and bounded. Then MOPVC-PMRC and MOPVC-PSRC hold at x^* .*

Proof. Suppose there is a nonempty set of proper multipliers at x^* , which is bounded. Also, assume that MOPVC-PMRC does not hold at x^* for the problem, that is, for all indices $r \in \mathcal{P}$, we have scalars $\alpha_i \geq 0$ ($i \in \mathcal{P}, i \neq r$), $\lambda_i \geq 0$, $\forall i \in I_g(x^*)$, $\lambda_i = 0 \quad \forall i \notin I_g(x^*)$

$$\eta_i^G = 0, \forall i \in I_{+-}(x^*) \cup I_{0-}(x^*) \cup I_{0+}(x^*), \eta_i^G \geq 0, \forall i \in I_{+0}(x^*) \cup I_{00}(x^*)$$

$$\text{and } \eta_i^H = 0, \forall i \in I_+(x^*), \eta_i^H \geq 0, \forall i \in I_{0-}(x^*) \text{ and } \eta_i^H \text{ is free } \forall i \in I_{0+}(x^*)$$

$$\eta_i^H \eta_i^G = 0, \forall i \in I_{00}(x^*)$$

not all zero, such that

$$(7) \quad 0 = \sum_{i \in \mathcal{P}, i \neq r} \alpha_i \nabla f_i(x^*) + \sum_{i=1}^l \lambda_i \nabla g_i(x^*) + \sum_{i=1}^m \mu_i \nabla h_i(x^*) \\ - \sum_{i=1}^k \eta_i^H \nabla H_i(x^*) + \sum_{i=1}^k \eta_i^G \nabla G_i(x^*)$$

Consider the set

$$L_r(x^*) = \{(\alpha, \lambda, \mu, \eta^G, \eta^H) \in \mathbb{R}^{p-1} \times \mathbb{R}_+^l \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \mid (\alpha, \lambda, \mu, \eta^G, \eta^H) \neq (0, 0, 0, 0, 0) \text{ and satisfies (7)}\}$$

Clearly, $L_r(x^*)$ is unbounded, because $sL_r(x^*) \subset L_r(x^*)$ for all $s > 0$. So for all indices $r \in \mathcal{P}$, the set $L_r(x^*)$ is unbounded.

It is given that there is a nonempty set of proper multipliers at x^* , suppose it corresponds to $r \in \mathcal{P}$, that is

$$L'_r(x^*) = \{(\alpha', \lambda', \mu', \eta^{G'}, \eta^{H'}) \mid \alpha'_r = 1 \text{ and (7) holds}\}$$

is the set of proper multipliers at x^* , which is bounded.

According to the assumption that MOPVC-PMRC does not hold at x^* , and using the structure of the above two sets of multipliers, we have

$$0 \in \nabla f_r(x^*) + \sum_{i \in \mathcal{P}, i \neq r} (\alpha'_i + s\alpha_i) \nabla f_i(x^*) + \sum_{i=1}^l (\lambda'_i + s\lambda_i) \nabla g_i(x^*) \\ + \sum_{i=1}^m (\mu'_i + s\mu_i) \nabla h_i(x^*) - \sum_{i=1}^k (\eta_i^{H'} + s\eta_i^H) \nabla H_i(x^*) + \sum_{i=1}^k (\eta_i^{G'} + s\eta_i^G) \nabla G_i(x^*)$$

for all $s > 0$ and

$$(\lambda'_i + s\lambda_i)g_i(x^*) = \lambda'_i g_i(x^*) + s\lambda_i g_i(x^*); \quad \forall i \in \mathcal{L}$$

Therefore

$$(\alpha'_1 + s\alpha_1, \dots, \alpha'_{r-1} + s\alpha_{r-1}, 1, \alpha'_{r+1} + s\alpha_{r+1}, \lambda'_1 + s\lambda_1, \dots, \lambda'_l + s\lambda_l, \mu'_1 + s\mu_1, \dots, \mu'_m + s\mu_m \\ \eta_1^{G'} + s\eta_1^G, \dots, \eta_k^{G'} + s\eta_k^G, \eta_1^{H'} + s\eta_1^H, \dots, \eta_k^{H'} + s\eta_k^H) \in L'_r(x^*); \quad \forall s > 0$$

Here, one of the coefficient must be nonzero for all $s > 0$, that is $L'_r(x^*)$ becomes unbounded. It is a contradiction to the assumption of the existence of a bounded set of multipliers. Hence, MOPVC-PMRC holds. Similarly, if a set of the proper multiplier is nonempty and bounded, then MOPVC-PSRC holds at x^* . ■

The following example shows that the set of proper multipliers is unbounded if MOPVC-PRC fails to hold.

Example 3.2. Consider the following MOPVC.

$$\min(f_1, f_2) \\ \text{where } f_1(x) = x_1^2 + x_3 \\ f_2(x) = x_3^2 + x_2 \\ g(x) = -x_2^2 + x_1 \leq 0 \\ h(x) = x_1 - x_2 = 0 \\ H(x) = x_1^3 + x_3 \geq 0 \\ \text{and } G(x)H(x) = -x_1(x_1^3 + x_3) \leq 0$$

Clearly, $x^* = (0,0,0)$ is an efficient solution to this problem. If $r = 1$, then

$$\alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \eta^G \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} - \eta^H \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \alpha_2 = 1, \quad \lambda = 1, \quad \mu = 1, \quad \eta^G = 2, \quad \eta^H = 0$$

If $r = 2$, then

$$\alpha_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \eta^G \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} - \eta^H \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \alpha_1 = \eta^H = 1, \quad \lambda = 1, \quad \mu = 0, \quad \eta^G = 1$$

That is, MOPVC-PRC fails to hold in both cases at $x^* = (0,0,0)$

Now, the set of multipliers for the index $r = 1$ and $r = 2$ respectively are

$$E_1(0,0,0) = \{(1, \alpha_2, \lambda, \mu, \eta^G, \eta^H)\}$$

such that

$$1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \eta^G \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} - \eta^H \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \eta^H = 1, \alpha_2 = \mu = \eta^G - \lambda$$

and

$$E_2(0,0,0) = \{(\alpha_1, 1, \lambda, \mu, \eta^G, \eta^H)\}$$

such that

$$\alpha_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \eta^G \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} - \eta^H \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \eta^H = \alpha_1, \mu = \eta^G - \lambda = 1$$

that is $E_1(0,0,0)$ and $E_2(0,0,0)$, Both are unbounded. ■

Now, based on these two results, Theorems 3.1 and 3.2, we are in a position to state the necessary and sufficient condition to ensure the boundedness of Lagrange multipliers.

Theorem 3.3. *Let x^* be a feasible point for smooth MOPVC. Then there exists a nonempty and bounded set of proper multipliers, if and only if at least one of MOPVC-PMRC and MOPVC-PSRC holds at x^* .*

4. Uniqueness of Lagrange multipliers for the multiobjective mathematical program with vanishing constraints

The question is natural about the necessity of the uniqueness of Lagrange multipliers. Ben-Tal¹⁵ and Kyparisis¹⁰ derived a second-order optimality condition under the unique Lagrange multiplier assumption. On the other

hand, we can see in the literature¹⁶⁻¹⁸ that the uniqueness of these multipliers decreases the difficulty in the study of sensitivity analysis and convergence analysis of the Newton-type optimization algorithm. In¹⁸, the author presented some approaches to studying the parametric program and investigated that these approaches are not well-suited when the set of Lagrange multipliers is not a singleton. Thus, we have enough motivation for investigating the uniqueness of KKT-type multipliers for MOPVC.

In problem (1), if $p = 1$ that is the problem is a single objective, then multiplier $(\alpha, \lambda, \mu, \eta^G, \eta^H)$ is said to be a KKT multiplier, if Theorem 2.1 holds with $\alpha = 1$. But in a multiobjective case, $\alpha = (\alpha_1, \dots, \alpha_p)$ is a vector, so we define $(\alpha, \lambda, \mu, \eta^G, \eta^H)$ as a KKT multiplier, if Theorem 2.1 holds with $\|\alpha\| = 1$.

Further, denote the set of KKT multipliers at x^* as

$$\mathcal{M}_+(x^*) = \{(\alpha, \lambda, \mu, \eta^G, \eta^H) \mid (3), (4) \text{ holds with } \|\alpha\| = 1\}.$$

Again, we recall sets $\mathcal{P}, \mathcal{L}, \mathcal{K}$ from section 2 and further partition them as follows.

$$\begin{aligned} \mathcal{P}_+ &= \{i \in \mathcal{P} : \alpha_i > 0\} \quad ; \quad \mathcal{P}_0 = \{i \in \mathcal{P} : \alpha_i = 0\} \\ I_g^+ &= \{i \in I_g(x) : \lambda_i > 0\} \quad ; \quad I_g^0 = \{i \in I_g(x) : \lambda_i = 0\} \\ I_G^+ &= \{i \in \mathcal{K} : \eta_i^G > 0\} \quad ; \quad I_G^0 = \{i \in \mathcal{K} : \eta_i^G = 0\} , \\ I_H^+ &= \{i \in \mathcal{K} : \eta_i^H > 0\} \quad ; \quad I_H^0 = \{i \in \mathcal{K} : \eta_i^H = 0\} \\ I_H^- &= \{i \in \mathcal{K} : \eta_i^H < 0\}. \end{aligned}$$

Now, based on these index sets, we introduce a new constraint regularity, which is inspired by⁹ as -

Definition 4.1. MOPVC-strict Mangasarian Fromovitz regularity condition (or MOPVC-SMFRC): At any feasible point $x^* \in \mathcal{X}$, MOPVC-SMFRC holds with respect to the index $s \in \mathcal{P}$, if for the index sets $\mathcal{P}_+, \mathcal{P}_0, I_g^+, I_g^0, I_G^+, I_G^0, I_H^+, I_H^0, I_H^-$ the following conditions hold.

1. $\nabla f_p(x^*), (p \in \mathcal{P}_+, p \neq s), \nabla g_i(x^*), (i \in I_g^+), \nabla h_i(x^*), (i \in \mathcal{M}), \nabla G_i(x^*), (i \in I_G^+), \nabla H_i(x^*), (i \in I_H^+)$ are linearly independent.
2. There exists a vector $d_s \in \mathbb{R}^n$ such that the following conditions hold.

$$\begin{aligned} \nabla f_p(x^*)^T d_s &< 0; p \notin \mathcal{P}_+, \quad p \neq s, \\ \nabla f_p(x^*)^T d_s &= 0; p \in \mathcal{P}_+, \quad p \neq s, \\ \nabla g_i(x^*)^T d_s &= 0; i \in I_g^+, \\ \nabla g_i(x^*)^T d_s &< 0; i \in I_g^0, \\ \nabla h_i(x^*)^T d_s &= 0; i \in \mathcal{M}, \end{aligned} \tag{8}$$

$$\begin{aligned}
\nabla G_i(x^*)^T d_s &= 0; i \in I_G^+, \\
\nabla G_i(x^*)^T d_s &< 0; i \in I_G^0, \\
\nabla H_i(x^*)^T d_s &= 0; i \in I_H^+ \cup I_H^-, \\
\nabla H_i(x^*)^T d_s &> 0; i \in I_H^0.
\end{aligned}$$

Before stating the main result of this section, we need the following lemma.

Lemma 4.1. *Let $(\alpha^*, \lambda^*, \mu^*, \eta^{G^*}, \eta^{H^*}) \in \mathcal{M}_+(x^*)$ be a KKT-multiplier for MOPVC (1) and system (8) admits solution for some $s \in \mathcal{P}_+$. If another multiplier $(\alpha', \lambda', \mu', \eta^{G'}, \eta^{H'}) \in \mathcal{M}_+(x^*)$, then*

$$\begin{aligned}
\{p \in \mathcal{P} : \alpha'_p > 0\} &\subseteq \mathcal{P}_+ \\
\{i \in \mathcal{L} : \lambda'_i > 0\} &\subseteq I_g^+ \\
\{i \in \mathcal{K} : \eta_i^{G'} > 0\} &\subseteq I_G^+ \\
\{i \in \mathcal{K} : \eta_i^{H'} \neq 0\} &\subseteq I_H^+ \cup I_H^-
\end{aligned}$$

Proof. Suppose there is a vector $d_s \in \mathbb{R}^n$, which solves the above system (8) for some $s \in \mathcal{P}_+$, then by Motzkin's alternative theorem¹⁹, there is no multiplier $(\alpha, \lambda, \mu, \eta^G, \eta^H) \neq (0, 0, 0, 0, 0)$ such that system (3) holds with $\alpha_k \geq 0$ ($i \notin \mathcal{P}_+$), $\lambda_i \geq 0$ ($i \in I_g^0$), $\eta_i^G \geq 0$ ($i \in I_G^0$) and $\eta_i^H \geq 0$ ($i \in I_H^0$).

Since $(\alpha', \lambda', \mu', \eta^{G'}, \eta^{H'})$ satisfies (3), therefore $\alpha'_p = 0$ ($p \notin \mathcal{P}_+$), $\lambda'_i = 0$ ($i \in I_g^0$),

$\eta_i^{G'} = 0$ ($i \in I_G^0$), $\eta_i^{H'} = 0$ ($i \in I_H^0$). Hence, the result is proved. ■

Here is the main result of this section.

Theorem 4.1. *Suppose that the KKT condition holds at $x^* \in \mathcal{X}$ for MOPVC with KKT multiplier $(\alpha^*, \lambda^*, \mu^*, \eta^{G^*}, \eta^{H^*}) \in \mathcal{M}_+(x^*)$. Then $\mathcal{M}_+(x^*)$ is a singleton if and only if MOPVC-SMFRC holds at x^* , for all $s \in \mathcal{P}_+$.*

Proof. Suppose $(\alpha^*, \lambda^*, \mu^*, \eta^{G^*}, \eta^{H^*}) \in \mathcal{M}_+(x^*)$ is unique, then we have to show that MOPVC-SMFRC holds. Suppose it does not hold, then either condition (1) or (2) of Definition 4.1 fails. In both cases, there exists $\alpha' \in \mathbb{R}^p$, $\lambda' \in \mathbb{R}^l$, $\mu' \in \mathbb{R}^m$, $\eta^{G'}, \eta^{H'} \in \mathbb{R}^n$, not all zero, such that $\alpha'_s = 0$, $\lambda'_i = 0$ ($i \notin \mathcal{L}$) and

$$\begin{aligned}
&\sum_{k \in \mathcal{P}} \alpha'_k \nabla f_k(x^*) + \sum_{i \in \mathcal{L}} \lambda'_i \nabla g_i(x^*) + \sum_{i \in \mathcal{M}} \mu'_i \nabla h_i(x^*) \\
&+ \sum_{i \in \mathcal{K}} \eta_i^{G'} \nabla G_i(x^*) - \sum_{i \in \mathcal{K}} \eta_i^{H'} \nabla H_i(x^*) = 0.
\end{aligned}$$

Without loss of generality, assume that

$$\max\{|\alpha'_k| : k \in \mathcal{P}_+\} < \min\{\alpha_k^* : k \in \mathcal{P}_+\}$$

$$\begin{aligned} \max\{|\lambda'_i| : i \in I_g^+\} &< \min\{\lambda_i^* : i \in I_g^+\} \\ \max\{|\eta_i^{G'}| : i \in I_G^+\} &< \min\{\eta_i^{G^*} : i \in I_G^+\}. \end{aligned}$$

then,

$$\begin{aligned} \alpha'_k + \alpha_k^* &\geq 0 ; (k \in \mathcal{P}), \\ \lambda'_i + \lambda_i^* &\geq 0 ; (i \in \mathcal{L}), \\ \eta_i^{G'} + \eta_i^{G^*} &\geq 0 ; (i \in \mathcal{K}), \\ \eta_i^{H'} + \eta_i^{H^*} &\in \mathbb{R} ; (i \in \mathcal{K}). \end{aligned}$$

Further, set $\|\alpha' + \alpha^*\|^{-1} = \gamma$, then we have

$$\gamma(\alpha' + \alpha^*, \lambda' + \lambda^*, \mu' + \mu^*, \eta^{G'} + \eta^{G^*}, \eta^{H'} + \eta^{H^*}) \in \mathcal{M}_+(x^*)$$

By the uniqueness assumption

$$\begin{aligned} \gamma(\alpha' + \alpha^*) &= \alpha^* \\ \Rightarrow \gamma\alpha'_k &= (1 - \gamma)\alpha_k^* ; k \in \mathcal{P}, \\ \text{similarly } \gamma\lambda'_i &= (1 - \gamma)\lambda_i^* ; i \in \mathcal{L}, \\ \gamma\eta_i^{G'} &= (1 - \gamma)\eta_i^{G^*} ; i \in \mathcal{K}, \\ \gamma\eta_i^{H'} &= (1 - \gamma)\eta_i^{H^*} ; i \in \mathcal{K}. \end{aligned}$$

Since $\alpha'_s = 0$ and $\alpha_s^* \neq 0$, therefore $1 - \gamma = 0 \Rightarrow \gamma = 1$. Then all the multipliers

$$\alpha' = \lambda' = \mu' = \eta^{G'} = \eta^{H'} = 0$$

It is a contradiction. So, MOPVC-SMFRC holds at x^* .

Conversely, suppose MOPVC-SMFRC holds at x^* . We have to prove $(\alpha^*, \lambda^*, \mu^*, \eta^{G'}, \eta^{H'}) \in \mathcal{M}_+(x^*)$ is unique. For this, let $(\alpha', \lambda', \mu', \eta^{G'}, \eta^{H'})$ be any other vector of $\mathcal{M}_+(x^*)$ and $s \in \mathcal{P}$ be such that $\alpha'_s \neq 0$.

By Lemma 4.1,

$$\begin{aligned} \alpha'_k &= 0 ; k \notin \mathcal{P}_+ \\ \lambda'_i &= 0 ; i \in I_g^0 \text{ or } i \notin I_g^+ \\ \eta_i^{G'} &= 0 ; i \in I_G^0 \text{ or } i \notin I_G^+ \\ \eta_i^{H'} &= 0 ; i \in I_H^0. \end{aligned}$$

So $\alpha'_s \neq 0$, therefore $s \in \mathcal{P}_+$ and we have

$$\begin{aligned} &\sum_{k \in \mathcal{P}_+} (\gamma\alpha'_k - \alpha_k^*) \nabla f_k(x^*) + \sum_{i \in I_g^+} (\gamma\lambda'_i - \lambda_i^*) \nabla g_i(x^*) + \\ (9) \quad &\sum_{i \in \mathcal{M}} (\gamma\mu'_i - \mu_i^*) \nabla h_i(x^*) + \sum_{i \in I_G^+} (\gamma\eta_i^{G'} - \eta_i^{G^*}) \nabla G_i(x^*) \\ &- \sum_{i \in I_H^+ \cup I_H^-} (\gamma\eta_i^{H'} - \eta_i^{H^*}) \nabla H_i(x^*) = 0 \end{aligned}$$

where $\gamma = \frac{\alpha_s^*}{\alpha_s'}$. Since $\gamma\alpha_s' - \alpha_s^* = 0$, therefore, all the coefficients in (9) must be zero because of the linearly independent restriction of MOPVC-SMFRC. So we can have

$$\gamma\alpha_k' - \alpha_k^* = 0 \Rightarrow \gamma\alpha_k' = \alpha_k^*.$$

Similarly $\gamma(\alpha', \lambda', \mu', \eta^{G'}, \eta^{H'}) = (\alpha^*, \lambda^*, \mu^*, \eta^{G^*}, \eta^{H^*})$. Moreover $1 = \|\alpha^*\| = \gamma\|\alpha'\| = \gamma$, so $(\alpha', \lambda', \mu', \eta^{G'}, \eta^{H'}) = (\alpha^*, \lambda^*, \mu^*, \eta^{G^*}, \eta^{H^*})$ The KKT multipliers are unique.

Conclusion: In this paper, we have studied the multiobjective optimization problem with vanishing constraints (MOPVC) with smoothness assumptions. There are two main analyses of this paper, the first is about the boundedness of certain sets of Lagrange multipliers, and the second is about the uniqueness of the same. Firstly, we have defined the proper M and S- regularity conditions for MOPVC and proved that these conditions are necessary and sufficient for the set of M- and S-multipliers to be nonempty and bounded. The second important work of this paper is to analyze the results of the uniqueness of the Lagrange multipliers. For this, we have introduced the MOPVC-strict Mangasarian Fromovitz regularity condition (or MOPVC-SMFCQ), which is motivated by the SMFRC of a standard multiobjective nonlinear program⁹. We have proved that this regularity condition is necessary and sufficient for KKT multipliers to be unique.

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