

Riemannian Maps Satisfying DDVV-Type Inequality

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Abstract: This paper establishes DDVV-type curvature inequalities for Riemannian maps into generalized complex space forms. Using the Gauss and Ricci equations, we relate intrinsic curvature invariants of the domain manifold to extrinsic invariants of the map and discuss the equality cases. The results extend known curvature inequalities from submanifold theory to the setting of Riemannian maps.

Keywords: Almost Hermitian manifold; generalized complex space form; Riemannian maps; DDVV conjecture.

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1. Introduction

In the field of geometry, inequalities serve to link intrinsic and extrinsic invariants, playing a crucial role in defining the characteristics of geometric objects. Among the most significant results in this area is the DDVV conjecture.

The DDVV conjecture has its roots in the earlier work of Pierre Wintgen¹, who in 1979 proved a seminal inequality for surfaces immersed in a 4-dimensional Euclidean space. This result was later generalized by Guadalupe and Rodriguez² to surfaces in arbitrary space forms. The DDVV conjecture, formally proposed in 1999 by De Smet, Dillen, Verstraelen, and Vrancken³,

extends this foundational work to higher dimensions and arbitrary codimension.

By a DDVV-type inequality, we mean an inequality that gives a bound of the form

$$(1.1) \quad \sum_{r,s=1}^m \|A_r, A_s\|^2 \leq c(\sum_{r=1}^m \|A_r\|^2)^2.$$

This inequality is formulated for a particular class of $n \times n$ matrices $A_1, \dots, A_m$, where the commutator of two matrices is defined by $[A, B] = AB - BA$. The quantity $\|A_r\|^2 = \text{tr}(AA^*)$ denotes the squared Frobenius norm, with A^* representing the conjugate transpose of A , and c being a non-negative real constant. Inequalities of DDVV type naturally arise from the normal scalar curvature problem in submanifold geometry, which is commonly known as the DDVV conjecture.

In 1999, De-Dillen-Verstraelen-Vrancken³ proposed the normal scalar curvature conjecture:

Conjecture: (DDVV conjecture) Let M^n be an immersed submanifold of a real space form with constant sectional curvature κ . Then

$$(1.2) \quad \rho + \rho^\perp \leq \|H\|^2 + \kappa,$$

where ρ denotes the normalized scalar curvature, which is an intrinsic geometric quantity determined by the curvature tensor of the tangent bundle, while $\rho^\perp$ represents the normalized normal scalar curvature associated with the normal bundle, an extrinsic invariant that captures the curvature of the normal bundle and H is the normalized mean curvature vector field, which measures the average extrinsic curvature of the submanifold.

The Normal Scalar Curvature Conjecture was proved by Lu⁴ and Ge-Tang⁵ independently. The submersion version of the normal scalar curvature conjecture was studied by Ge⁶.

The DDVV conjecture, as originally stated and proven, applies to submanifolds of real space forms. The problem of generalizing this inequality to more intricate ambient spaces and types of maps has become a significant area of contemporary research. The DDVV inequality has been successfully extended to submanifolds of warped products, as well as to special classes of submanifolds, such as slant, totally real, and CR-submanifolds, particularly in complex and quaternionic ambient spaces.

Recall that a Riemannian map was defined by Fischer⁷ to be a mapping whose differential acts as an isometry from the tangent space of the domain, modulo its kernel, onto its image. A Riemannian map is a smooth map between Riemannian manifolds that generalizes both isometric immersions and Riemannian submersions. Unlike an isometric immersion, which has an injective differential, a Riemannian map can have a non-trivial kernel (or vertical distribution) at each point.

Fischer outlined in his seminal work⁷ that a Riemannian map "is a map that is as isometric as it can be, subject to the limitations imposed upon it as a differentiable mapping". He demonstrated that these maps fulfill the generalized eikonal equation and thus have constant rank on each connected component, establishing a connection between physical optics and geometric optics.

Moreover, Fischer proposed that Riemannian maps can serve as a geometric framework for formulating quantum models, thereby relating the theory of Riemannian and harmonic maps to Lagrangian field theory and to physical models governed by the Maxwell-Schrödinger equations.

The structure of the paper is outlined as follows. Section 2 revisits the fundamental concepts and formulas necessary for the discussion. In Section 3, we explore the DDVV inequality for Riemannian maps transitioning from a Riemannian manifold to a generalized complex space form. Lastly, the paper concludes with the application of such inequalities.

2. Preliminaries

In this section, we recall some fundamental notions and preliminary results related to the geometric framework of Riemannian maps, highlighting the role of the second fundamental form.

2.1 Generalized Complex Space Form

Definition 2.1⁸⁻¹¹ Let (M, g_1, J) be a Riemannian manifold of even-dimension equipped with a structure J which is almost complex of type $(1,1)$ on M satisfying $J^2 = -I$, where I denotes the identity map. The manifold (M, g_1, J) is called an almost Hermitian manifold if the Riemannian metric g_1 is compatible with J , that is,

$$g_1(JX, JY) = g_1(X, Y) \text{ for all } X, Y \in \Gamma(TM).$$

Moreover, (M, g_1, J) is said to be a generalized complex space form, denoted by $M(c_1, c_2)$, if there exist smooth functions c_1 and c_2 on M such that the curvature tensor R of M satisfies, for all $X, Y, Z \in \Gamma(TM)$,

$$(2.1) \quad R(X, Y)Z = c_1\{g_1(Y, Z)X - g_1(X, Z)Y\} \\ + c_2\{g_1(JY, Z)JX - g_1(JX, Z)JY + 2g_1(JY, X)JZ\}.$$

In particular, the following special cases are well known¹⁰⁻¹²:

- If $c_1 = c$ and $c_2 = 0$, then $M(c)$ reduces to a real space form.
- If $c_1 = c_2 = \frac{c}{4}$, then $M(c)$ is a complex space form.
- If $c_1 = \frac{c+3\alpha}{4}$ and $c_2 = \frac{c-\alpha}{4}$, then $M(c)$ becomes a real Kähler space form.

2.2 Riemannian maps

Let $F: (M, g_1) \rightarrow (N, g_2)$ be a smooth mapping between Riemannian manifolds M and N of dimensions m and n , respectively, such that

$$0 < \text{rank} F < \min\{m, n\}.$$

For each point $p \in M$, the differential of F at p is denoted by

$$F_{*p}: T_p M \rightarrow T_{F(p)} N.$$

The kernel of F_{*p} determines an orthogonal decomposition of the tangent space $T_p M$ with respect to the metric g_1 given by⁷

$$T_p M = \ker F_{*p} \oplus (\ker F_{*p})^\perp$$

The subspaces $\ker F_{*p}$ and $(\ker F_{*p})^\perp$ are referred to as the vertical and horizontal distributions of M at p , respectively, and are denoted by $\mathcal{V}_p$ and $\mathcal{H}_p$. Since the rank of F is neither zero nor maximal, the image of F_{*p} does not coincide with the entire tangent space of N at $F(p)$. Consequently, the tangent space $T_{F(p)} N$ admits the orthogonal decomposition

$$T_{F(p)} N = \text{range} F_{*p} \oplus (\text{range} F_{*p})^\perp$$

with respect to the metric g_2 .

The map F is called a *Riemannian map* if for each point $p \in M$, the restriction

$$F_{*p}|_{(\ker F_{*p})^\perp}: (\ker F_{*p})^\perp \rightarrow \text{range} F_{*p}$$

is a linear isometry. Equivalently,

$$g_2(F_*X, F_*Y) = g_1(X, Y)$$

for all vector fields $X, Y \in \Gamma((\ker F_*)^\perp)$. This notion unifies two classical geometric mappings. In particular, when $\ker F_* = \{0\}$, the map F reduces to an isometric immersion, while if $(\text{range} F_*)^\perp = \{0\}$, it becomes a Riemannian submersion.

To describe the extrinsic geometry associated with the map F , the second fundamental form of F is defined by¹³

$$(\nabla F_*)(X, Y) = \nabla_X^N F_*Y - F_*(\nabla_X^M Y)$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$, where ∇^M and ∇^N denote the Levi-Civita connections on M and N , respectively. Here, $(\nabla F_*)(X, Y)$ takes values in $(\text{range} F_*)^\perp$.

Let $\nabla^{F^\perp}$ be the induced connection on the normal bundle $(\text{range} F_*)^\perp$. For any $X \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma((\text{range} F_*)^\perp)$, the covariant derivative $\nabla_{F_*X}^N V$ can be decomposed as¹³

$$\nabla_{F_*X}^N V = -S_V F_*X + \nabla_X^{F^\perp} V,$$

where S_V denotes the shape operator corresponding to the normal vector field V . The shape operator satisfies the relation

$$(2.2) \quad g_2(S_V F_* X, F_* Y) = g_2(V, (\nabla F_*)(X, Y)),$$

for all $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma((\text{range } F_*)^\perp)$. Since the second fundamental form (∇F_*) is symmetric, it follows that each shape operator S_V is a symmetric linear transformation of $\text{range } F_*$.

Let R^M, R^N and $R^\perp$ denote the curvature tensors corresponding to the Levi-Civita connections ∇^M on M, ∇^N on N , and the normal connection on $(\text{range } F_*)^\perp$, respectively. For a Riemannian map F , the Gauss and Ricci equations relate the curvature tensors of M and N to the second fundamental form and the shape operators. These equations are given as follows¹³:

The Gauss equation for a Riemannian map is expressed by

$$(2.3) \quad g_2(R^N(F_* X, F_* Y)F_* Z, F_* H) = g_1(R^M(X, Y)Z, H) \\ + g_2((\nabla F_*)(X, Z), (\nabla F_*)(Y, H)) - g_2((\nabla F_*)(Y, Z), (\nabla F_*)(X, H))$$

for all vector fields $X, Y, Z, H \in \Gamma((\ker F_*)^\perp)$.

Similarly, the Ricci equation associated with the normal bundle $(\text{range } F_*)^\perp$ is given by

$$(2.4) \quad g_2(R^N(F_* X, F_* Y)V_1, V_2) = g_2(R^{F^\perp}(F_* X, F_* Y)V_1, V_2) + g_2([S_{V_2}, S_{V_1}]F_* X, F_* Y)$$

where $V_1, V_2 \in \Gamma((\text{range } F_*)^\perp)$ and the commutator of the shape operators is defined by

$$[S_{V_2}, S_{V_1}] = S_{V_2}S_{V_1} - S_{V_1}S_{V_2}$$

Let $\{e_1, \dots, e_r\}$ be an orthonormal basis of $(\ker F_*)_p^\perp$ and $\{v_{r+1}, \dots, v_n\}$ an orthonormal basis of $(\text{range } F_*)_p^\perp$ at a point $p \in M$. The Ricci curvature of the horizontal distribution $(\ker F_*)_p^\perp$ is then defined by

$$\text{Ric}^{(\ker F_*)^\perp}(X) = \sum_{i=1}^r g_1(R^M(e_i, X)X, e_i), X \in (\ker F_*)^\perp.$$

The corresponding scalar curvature is given by

$$\tau^{(\ker F_*)^\perp} = \sum_{1 \leq i < j \leq r} g_1(R^M(e_i, e_j)e_j, e_i)$$

and the normalized scalar curvature is defined as

$$\varrho^{(\ker F_*)^\perp} = \frac{2\tau^{(\ker F_*)^\perp}}{r(r-1)}$$

On the other hand, the scalar curvature of the normal subbundle $(\text{range } F_*)^\perp$, referred to as the normal scalar curvature, is defined by

$$\tau^{(\text{range } F_*)^\perp} = \sqrt{\sum_{1 \leq i < j \leq r} \sum_{r+1 \leq \beta < \gamma \leq n} g_2(R^\perp(e_i, e_j)v_\beta, v_\gamma)^2}$$

The corresponding normalized normal scalar curvature is given by

$$(2.5) \quad \varrho^{(\text{range } F_*)^\perp} = \frac{\tau^{(\text{range } F_*)^\perp}}{r(r-1)}$$

Also, for the Riemannian map F , we can write¹⁴

$$(2.6) \quad \zeta_{ij}^\beta = g_2 \left((\nabla F_*)(e_i, e_j), v_\beta \right), i, j = 1 \dots, r, \beta = r+1, \dots, n$$

$$(2.7) \quad \|\zeta\|^2 = \sum_{i,j=1}^r g_2 \left((\nabla F_*)(e_i, e_j), (\nabla F_*)(e_i, e_j) \right),$$

$$(2.8) \quad \text{trace} \zeta = \sum_{i=1}^r (\nabla F_*)(e_i, e_j),$$

$$(2.9) \quad \|\text{trace} \zeta\|^2 = g_2(\text{trace} \zeta, \text{trace} \zeta).$$

Therefore, we have

$$(2.10) \quad \mathcal{C}^{\mathcal{H}} = \frac{1}{r} \|\zeta\|^2,$$

which is called the *Casorati curvature* of the horizontal space $\mathcal{H}^p$.

Lemma 2.2^{14,15}: For the above terms, we have

$$(2.11) \quad r\mathcal{C}^{\mathcal{H}} = \|\zeta\|^2 = \frac{1}{2} \|\text{trace} \zeta\|^2 + \frac{1}{2} \sum_{\beta=1}^n \left(\zeta_{11}^\beta - \zeta_{22}^\beta - \dots - \zeta_{rr}^\beta \right)^2 \\ + 2 \sum_{\beta=r+1}^n \sum_{i=2}^r \left(\zeta_{1i}^\beta \right)^2 - 2 \sum_{\beta=r+1}^n \sum_{2 \leq i < j \leq r} \left\{ \zeta_{ii}^\beta \zeta_{jj}^\beta - \left(\zeta_{ij}^\beta \right)^2 \right\}.$$

Lemma 2.3⁴: Let $\left(\zeta_{ij}^\beta \right), i, j = 1, \dots, r$ and $\beta = 1, \dots, n$ represent the coefficients of the traceless part of the second fundamental form relative to orthonormal bases of the tangent and normal bundles. Then

$$(2.12) \quad \sum_{\beta=1}^n \sum_{1 \leq i < j \leq r} \left(\zeta_{ii}^\beta - \zeta_{jj}^\beta \right)^2 + 2r \sum_{\beta=1}^n \sum_{1 \leq i < j \leq r} \left(\zeta_{ij}^\beta \right)^2 \\ \geq 2r \left(\sum_{1 \leq \beta < \gamma \leq n} \sum_{1 \leq i < j \leq r} \left(\sum_{k=1}^r \left(\zeta_{jk}^\beta \zeta_{ik}^\gamma - \zeta_{ik}^\beta \zeta_{jk}^\gamma \right) \right)^2 \right)^{\frac{1}{2}}.$$

3. Riemannian Maps to Generalized Space Forms

Let $\{e_1, \dots, e_r\}$ be an orthonormal basis of $(\ker F_*)^\perp_p$ and let $\{v_{r+1}, \dots, v_n\}$ be an orthonormal basis of $(\text{range } F_*)^\perp_{F_*p}$, where $p \in M$. Then $\{F_*e_1, \dots, F_*e_r\}$ forms an orthonormal basis of $(\text{range } F_*)_{F_*p}$.

Theorem 3.1: Assume that $\text{rank } F = r < n$ and let $F: (M, g_1) \rightarrow (M(c_1, c_2), g_2)$ be a Riemannian map defined from a Riemannian manifold (M, g_1) to a generalized complex space form $M(c_1, c_2)$. For a unit vector $X \in (\ker F_*)^\perp$, the inequality

$$(3.1) \quad 4\mathcal{R}ic^{(\ker F_*)^\perp}(X) \leq 4c_1(r-1) + \|\text{trace} \zeta\|^2 + 12c_2 \sum_{i=1}^r g_2^2(JF_*X, F_*e_i)$$

holds. Moreover, equality occurs if and only if

$$(3.2) \quad (\nabla F_*)(X, Y) = 0 \quad \forall Y \in \Gamma((\ker F_*)^\perp) \text{ orthogonal to } X,$$

and

$$(3.3) \quad (\nabla F_*)(X, X) = \frac{1}{2} \text{trace} \zeta.$$

Proof. Let $M(c_1, c_2)$ be a generalized space form with constant sectional curvature c_1 and c_2 , then from (2.3) and (2.4), we have

$$(3.4) \quad g_1(R^M(X, Y)Z, H) = c_1\{g_1(Y, Z)g_1(X, H) - g_1(X, Z)g_1(Y, H)\} \\ - g_2((\nabla F_*)(X, Z), (\nabla F_*)(Y, H)) + g_2((\nabla F_*)(Y, Z), (\nabla F_*)(X, H)) \\ + c_2\{g_2(JF_*Y, F_*Z)g_2(JF_*X, F_*H) - g_2(JF_*X, F_*Z)g_2(JF_*Y, F_*H) \\ + 2g_2(JF_*Y, F_*X)g_2(JF_*Z, F_*H)\},$$

where $X, Y, Z, H \in \Gamma((\ker F_*)^\perp)$. Let $\{e_1, \dots, e_r\}$ with $e_1 = X$ and using (2.6)-(2.9) and (3.4), we get

$$(3.5) \quad c_1 r(r-1) + 3c_2 \sum_{i,j=1}^r g_2^2(JF_*e_i, F_*e_j) = \|\zeta\|^2 - \|\text{trace} \zeta\|^2 + 2\tau^{(\ker F_*)^\perp},$$

using lemma (2.2), and

$$(3.6) \quad \tau^{(\ker F_*)^\perp}(p) = \frac{c_1}{2}(r-1)(r-2) + 3c_2 \sum_{2 \leq i < j \leq r} g_2^2(JF_*e_i, F_*e_j) \\ - \sum_{\beta=r+1}^n \sum_{2 \leq i < j \leq r} \left\{ (\zeta_{ij}^\beta)^2 - \zeta_{ii}^\beta \zeta_{jj}^\beta \right\} + \mathcal{Ric}^{(\ker F_*)^\perp}(X).$$

Substituting the value of (2.11) and (3.6) in (3.5), we get

$$(3.7) \quad \mathcal{Ric}^{(\ker F_*)^\perp}(X) = c_1(r-1) + \frac{1}{4} \|\text{trace} \zeta\|^2 + 3c_2 \sum_{i=1}^r g_2^2(JF_*X, F_*e_i) \\ - \frac{1}{4} \sum_{\beta=r+1}^n \left(\zeta_{11}^\beta - \zeta_{22}^\beta - \dots - \zeta_{rr}^\beta \right)^2 - \sum_{\beta=r+1}^n \sum_{i=2}^r \left(\zeta_{1i}^\beta \right)^2 \\ \leq c_1(r-1) + \frac{1}{4} \|\text{trace} \zeta\|^2 + 3c_2 \sum_{i=1}^r g_2^2(JF_*X, F_*e_i).$$

Hence, we have the inequality (3.1). The equality case of (3.1) holds if and only if

$$\zeta_{11}^\beta = \zeta_{22}^\beta + \dots + \zeta_{rr}^\beta$$

and

$$\zeta_{1i}^\beta = 0, i = 2, \dots, r; \beta = r+1, \dots, n;$$

hence we have (3.2).

Theorem 3.2: Assume that $\text{rank } F = r < n$ and let $F: (M, g_1) \rightarrow (M(c_1, c_2), g_2)$ be a Riemannian map defined from a Riemannian manifold (M, g_1) to a generalized complex space form $M(c_1, c_2)$. Then the following inequality holds

$$(3.8) \quad \varrho^{(\text{range } F_*)^\perp} + \varrho^{(\ker F_*)^\perp} \leq \frac{1}{r^2} \|\text{trace}\|^2 + c_1 + \frac{3c_2}{r(r-1)} \sum_{i,j=1}^r g_2^2(JF_*e_i, F_*e_j),$$

for normalized normal scalar curvature $\varrho^{(\text{range } F_*)^\perp}$ and normalized scalar curvature $\varrho^{(\ker F_*)^\perp}$.

Proof. Let $\{e_1, \dots, e_r\}$ be the orthonormal basis of $(\ker F_*)^\perp_p$ and $\{v_{r+1}, \dots, v_n\}$ be an orthonormal basis of $(\text{range } F_*)^\perp_{F_*p}$, where $p \in M$. Then $\{F_*e_1, \dots, F_*e_r\}$ forms the orthonormal basis of $(\text{range } F_*)_{F_*p}$, and from (2.1), (2.3) and (2.4), we have

$$(3.9) \quad \sum_{\beta, \gamma=r+1}^n \sum_{i,j=1}^r g_2(R^{F^\perp}(F_*e_i, F_*e_j)v_\beta, v_\gamma) \\ = g_2(R^N(F_*e_i, F_*e_j)v_\beta, v_\gamma) - g_2\left([S_{v_\gamma}, S_{v_\beta}]F_*e_i, F_*e_j\right),$$

using (3.9) in (2.5), we get

$$(3.10) \quad \varrho^{(\text{range } F_*)^\perp} = \frac{2}{r(r-1)} \sqrt{\sum_{1 \leq i < j \leq r} \sum_{1 \leq \beta < \gamma \leq n} g_2^2\left([S_{v_\gamma}, S_{v_\beta}]F_*e_i, F_*e_j\right)},$$

using (2.2) in (3.10) and solving further, we get

$$(3.11) \quad \varrho^{(\text{range } F_*)^\perp} = \frac{2}{r(r-1)} \sqrt{\sum_{1 \leq i < j \leq r} \sum_{1 \leq \beta < \gamma \leq n} \left(\sum_{k=1}^r (\zeta_{jk}^\beta \zeta_{ik}^\gamma - \zeta_{ik}^\beta \zeta_{jk}^\gamma) \right)^2},$$

using lemma (2.3), in (3.11), we get

$$(3.12) \quad r^2(r-1)\varrho^{(\text{range } F_*)^\perp} \leq \sum_{\beta=r+1}^n \sum_{1 \leq i < j \leq r} (\zeta_{ii}^\beta - \zeta_{jj}^\beta)^2 \\ + 2r \sum_{\beta=r+1}^n \sum_{1 \leq i < j \leq r} (\zeta_{ij}^\beta)^2.$$

(Remark- $\zeta_{ii}^\beta = 0$ for $\beta = 1, \dots, r$, from equation (2.6))

Also, we can compute

$$(3.13) \quad (r-1)\|\text{trace}\zeta\|^2 = \sum_{\beta=r+1}^n \sum_{1 \leq i < j \leq r} \left((\zeta_{ii}^\beta - \zeta_{jj}^\beta)^2 + 2r\zeta_{ii}^\beta \zeta_{jj}^\beta \right),$$

and

$$(3.14) \quad \tau^{(\ker F_*)^\perp} = r-1 \left(\frac{c_1}{2} r \right) + \frac{3c_2}{2} \sum_{i,j=1}^r g_2^2(JF_*e_i, F_*e_j) \\ + \sum_{\beta=r+1}^n \sum_{1 \leq i < j \leq r} (\zeta_{ii}^\beta \zeta_{jj}^\beta - (\zeta_{ij}^\beta)^2).$$

From (3.12), (3.13) and (3.14), we get the required result.

Remark: The establishment of a DDVV-type inequality for Riemannian maps from a Riemannian manifold into a generalized space form represents a significant advancement in differential geometry. The core application of the DDVV inequality for Riemannian maps is to establish a relationship between the curvature of the source manifold and the extrinsic curvature of the map.

They serve as a fundamental way to measure how a map "bends" or "curves" as it projects one manifold onto another. When the inequality becomes an equality, it characterizes a specific class of maps with special geometric properties. These maps are considered "ideal" or "optimal" in some geometric sense and can lead to rigidity results, which state that a map satisfying the equality must be of a very specific, rigid type. This work provides a unifying framework that bridges several distinct and complex areas of research: the DDVV conjecture, and the generalized mapping theory of Riemannian maps.

The results of this paper can be applied to a variety of related fields. In mathematical physics, complex geometric structures play a significant role in areas such as gauge theory, sigma models, and complex geometric formulations of field theories. Riemannian maps and submersions are used to model phenomena in Kaluza-Klein theory, where higher-dimensional manifolds are projected onto lower-dimensional ones. The DDVV-type inequality can provide new analytical tools for these applications, offering constraints and relationships between physical quantities that are geometrically motivated.

Declarations:

Data Availability: No new data were created or analysed in this study.

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