

Ricci Solitons on Para-Kenmotsu Manifolds Admitting Schouten-Van-Kampen Connection

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Abstract: In this paper, we study para-Kenmotsu manifolds equipped with the Schouten-Van-Kampen connection in the framework of Ricci solitons geometry. We begin by analyzing curvature tensor associated with this connection and establish several intrinsic properties of para-kenmotsu manifolds. Ricci solitons compatible with Schouten-van-Kampen connection are then characterized, yielding new geometric insights. Furthermore, we investigate specific curvature condition under which such manifolds admit Ricci solitons, with particular attention given to the projective flatness and concurrency of the structure vector field ξ . Finally, the existence and behaviour of Ricci solitons on submanifolds of para-Kenmotsu manifolds are examined relative to the Levi-Civita connection and Schouten-Van-Kampen connection respectively.

Keywords: Para-Kenmotsu manifolds, Ricci solitons, Schouten-Van-Kampen connection, curvature tensors, Submanifolds.

MSC 2020: 53C15, 53C25, 53D15.

1. Introduction

The geometry of almost contact manifolds equipped with pseudo-Riemannian metrics has been a subject of sustained interest in differential geometry. The foundational study in this direction was initiated by Takahashi¹ in 1969. Subsequently, Sato² introduced the notion of almost para-contact Riemannian manifolds in 1976, thereby opening new avenues in para-contact geometry. Further, developments were carried out by Adati and Mastsumoto³, who characterized P -Sasakian manifolds. In 2008, U.C.

De⁴ studied ϕ -Symmetric Kenmotsu manifolds while, the geometric aspects of Kenmotsu manifolds were studied by Singh *et.al.*⁵. Among the various classes of almost para contact metric manifolds, para-kenmotsu manifolds form a significant and well-structured category. These manifolds arise as a natural para-analogue of Kenmotsu manifolds in contact geometry and exhibit rich curvature and structural properties studied by Kenmotsu⁶. The concept of para-Kenmotsu manifolds was formally introduced by Welyczko⁷, who established their basic geometric framework. Significant characterizations of para-Kenmotsu manifolds and their special cases have been established by Sinha *et.al.*⁸, Blaga⁹ and Sai Prasad *et.al.*¹⁰. Subsequently, para-Kenmotsu manifolds have been extensively investigated over an extended period by several researchers, including Raghuwanshi *et al.*¹¹, Sardar¹². In 1982, Hamilton¹³ introduced the fundamental concept of Ricci solitons, which can be regarded as a natural generalization of Einstein metrics on Riemannian manifolds (M^n, g) . A Ricci soliton is defined by

$$(1.1) \quad (\mathcal{L}_V g)(X_1, Y_1) + 2S(X_1, Y_1) + 2\lambda g(X_1, Y_1) = 0,$$

where $\mathcal{L}_V$ denotes Lie derivative operator along the vector field V , where V is a potential vector field, S is the Ricci tensor g and λ is the Riemannian metric and scalar respectively. A Ricci soliton (M^n, g, V, λ) is shrinking soliton, steady soliton and expanding soliton according to λ (constant nature), *i.e.* λ is negative, zero and positive, respectively. U.C. De *et al.*¹⁴ explored Ricci solitons in a Kenmotsu manifold. The geometric aspects of Ricci solitons have been widely studied by numerous researchers in the context of contact and Lorentzian manifolds, including Bagewadi *et al.*¹⁵, Patel *et al.*¹⁶, Pandey *et al.*¹⁷, Ricci solitons and gradient Ricci solitons on $N(k)$ para-contact manifolds studied by De *et al.*¹⁸ and Siddiqui *et al.*¹⁹.

Among several affine connection, the Schouten-Van-Kampen connection occupies a prominent position owing to its natural compatibility with complementary distributions. This connection was introduced in the early twentieth century in the context of non-holonomic geometry and has since been recognized as an effective tool for studying manifolds endowed with special geometric structure. It is particularly well adapted to manifolds equipped with an affine connection and a pair of mutually complementary distributions. Let M^n be a differentiable manifold endowed with a Riemannian metric g , a vector field ξ and 1-form η . Consider the distributions $H = \ker \eta$, $V = \text{span } \xi$. Then $TM = H \oplus V$, with respect to Levi-Civita connection ∇ , The Schouten-Van-Kampen connection is defined by Olszak²⁰

$$(1.2) \quad \nabla_{X_1}^* Y_1 = \nabla_{X_1} Y_1 - \eta(Y_1) \nabla_{X_1} \xi + (\nabla_{X_1} \eta) Y_1 \xi.$$

This connection preserves the fundamental geometric object of the manifolds and satisfies the conditions. The geometric significance of the Schouten-Van-Kampen connection was first highlighted by Solvev²¹, who investigated hyperdistributions in Riemannian using this connection. Later, Bejancu²² extended these ideas and contributed further results in this direction. In this direction, Olszak²⁰ investigated the adaptation of the Schouten-Van-Kampen connection to almost contact metric manifolds and characterized several classes of such manifolds admitting this connection. Subsequently, the geometric properties of Kenmotsu manifolds equipped with the Schouten-Van-Kampen connection were explored by several authors. Notably, Gurupaddva Ingalahalli *et al.*²³ derived important results on Kenmotsu manifolds, while B. Kumhar *et al.*²⁴ extended these investigations to ϵ -Kenmotsu manifolds admitting Schouten-Van-Kampen connection.

The article is structured as follows: after the introductory section, section 2 is devoted to a concise review of the fundamental definition and essential properties of para-Kenmotsu manifolds, which form the geometric foundation of the present study. In section 3, we establish several curvature tensor identities with respect to the Schouten-Van-Kampen connection on para-Kenmotsu manifolds. Section 4, is concerned with the analysis of Ricci solitons on para-Kenmotsu manifolds admitting the Schouten-Van-Kampen connection. In section 5, we obtain further curvature results associated with Ricci solitons in this setting. The analysis of Ricci solitons on submanifolds of para-Kenmotsu manifolds admitting the Levi-Civita connection is presented in section 6. Finally, the last section addressed the behaviour of Ricci solitons on submanifolds of para-Kenmotsu manifolds with respect to Schouten-Van-Kampen connection.

2. Para-Kenmotsu Manifolds

An n -dimensional Pseudo-Riemannian manifold (M^n, g) is said to be an almost para contact manifold with an almost contact structure (ϕ, ξ, η, g) if there exists a (1,1)-tensor field ϕ , a vector field ξ and a 1-form η s.t.

$$(2.1) \quad \phi^2 X_1 = X_1 - \eta(X_1)\xi, \eta(\xi) = 1,$$

$$(2.2) \quad g(\phi X_1, \phi Y_1) = -g(X_1 Y_1) + \eta(X_1)\eta(Y_1),$$

for all vector fields X_1, Y_1 on M^n . In this case, we have $\phi\xi = 0$, $\eta\phi = 0$ and $\eta(X_1) = g(X_1, \xi)$. From equation (2.2), it can be easily deduced that $g(X_1, \phi Y_1) = -g(\phi X_1, Y_1)$ for all vector fields X_1, Y_1 on M^n . An almost para contact manifold M^n is called a para Kenmotsu manifold if

$$(2.3) \quad (\nabla_{X_1} \phi)Y_1 = g(X_1, \phi Y_1)\xi - \eta(Y_1)\phi X_1,$$

for all vector fields X_1, Y_1 on M^n , where ∇ is the Levi-Civita connection with respect to metric g . In a para-Kenmotsu manifold, we have

$$(2.4) \quad \nabla_{X_1} \xi = X_1 - \eta(X_1)\xi,$$

$$(2.5) \quad (\nabla_{X_1} \eta)Y_1 = g(X_1, Y_1) - \eta(X_1)\eta(Y_1).$$

Using equations (2.4) and (2.5), we find

$$(2.6) \quad R(X_1, Y_1)\xi = \eta(X_1)Y_1 - \eta(Y_1)X_1,$$

$$(2.7) \quad R(\xi, X_1)Y_1 = -g(X_1, Y_1)\xi + \eta(Y_1)X_1.$$

$$(2.8) \quad \eta(R(X_1, Y_1)Z_1) = g(X_1, Z_1)\eta(Y_1) - g(Y_1, Z_1)\eta(X_1),$$

for all vector fields X_1, Y_1, Z_1 , where R is the Riemann curvature tensor, S is the Ricci tensor of a para Kenmotsu manifold M^n .

$$(2.9) \quad S(X_1, \xi) = -(n-1)\eta(X_1),$$

for any vector field X_1 on M^n .

3. Curvature Tensor on Schouten-Van-Kampen Connection of Para-Kenmotsu Manifolds

The Schouten-Van-Kampen connection ∇^* affiliated to the Levi-Civita connection ∇ is endowed by

$$(3.1) \quad \nabla_{X_1}^* Y_1 = \nabla_{X_1} Y_1 - \eta(Y_1)\nabla_{X_1} \xi + (\nabla_{X_1} \eta)Y_1 \xi,$$

for any vector fields X_1, Y_1 on M^n .

Again using equations (2.4) and (2.5) in equation (3.1)

$$(3.2) \quad \nabla_{X_1}^* Y_1 = \nabla_{X_1} Y_1 - \eta(Y_1)X_1 + g(X_1, Y_1)\xi.$$

Putting $Y_1 = \xi$ in equation (3.2), we get

$$(3.3) \quad \nabla_{X_1}^* \xi = 0.$$

Suppose R and R^* are the curvature tensor with respect to ∇ and ∇^* , Levi-Civita connection and Schouten-Van-Kampen connection respectively, then

$$(3.4) \quad R^*(X_1, Y_1)Z_1 = \nabla_{X_1}^* \nabla_{Y_1}^* Z_1 - \nabla_{Y_1}^* \nabla_{X_1}^* Z_1 - \nabla_{[X_1, Y_1]}^* Z_1.$$

Again, by equation (3.4), we have

$$(3.5) \quad R^*(X_1, Y_1)Z_1 = R(X_1, Y_1)Z_1 - g(X_1, Z_1)Y_1 + g(Y_1, Z_1)X_1,$$

$$(3.6) \quad R^*(X_1, Y_1, Z_1, W_1) = R(X_1, Y_1, Z_1, W_1) - g(X_1, Z_1)g(Y_1, W_1) + g(Y_1, Z_1)g(X_1, W_1),$$

$$(3.7) \quad S^*(Y_1, Z_1) = S(Y_1, Z_1) + (n-1)g(Y_1, Z_1).$$

Again, contracting equation (3.7), we get

$$(3.8) \quad r^* = r + (n-1).$$

By equation (3.5), if $Z_1 = \xi$, then

$$(3.9) \quad R^*(X_1, Y_1)\xi = 0,$$

from equation (3.7), we have

$$(3.10) \quad S^*(Y_1, \xi) = S(Y_1, \xi) + (n-1)\eta(Y_1),$$

$$(3.11) \quad S^*(Y_1, \xi) = 0.$$

Again, by the first Bianchi identity and equation (3.5), we arrive at

$$(3.12) \quad R^*(X_1, Y_1)Z_1 + R^*(Y_1, Z_1)X_1 + R^*(Z_1, X_1)Y_1 = 0.$$

So, we can state that

Theorem 3.1: *On a Schouten-Van-Kampen connection of para Kenmotsu manifolds, then*

1. *the Riemannian curvature tensor is assigned by the equation (3.5),*
2. *the Ricci tensor is determined through the equation (3.7),*
3. *the expression for the scalar curvature tensor is given by equation (3.8),*
4. *M^n satisfies the first Bianchi identity given by the equation (3.12).*

4. Ricci solitons on Para Kenmotsu manifolds with respect to Schouten-Van-Kampen connection

A Ricci soliton on (g, V, λ) on para Kenmotsu manifolds admits Schouten-Van-Kampen connection is defined by

$$(4.1) \quad (\mathcal{L}_V^*g)(Y_1, Z_1) + 2S^*(Y_1, Z_1) + 2\lambda g(Y_1, Z_1) = 0,$$

we have

$$(4.2) \quad (\mathcal{L}_V^*g)(Y_1, Z_1) = g(\nabla_{Y_1}^*V, Z_1) + g(Y_1, \nabla_{Z_1}^*V).$$

By equation (3.2), we obtain

$$(4.3) \quad (\mathcal{L}_V^*g)(Y_1, Z_1) = (\mathcal{L}_Vg)(Y_1, Z_1) - 2\eta(V)g(Y_1, Z_1) \\ + g(Y_1, V)\eta(Z_1) + g(Z_1, V)\eta(Y_1).$$

Again by using equations (3.7), (4.3) and (4.1), we get

$$(4.4) \quad -2\eta(V)g(Y_1, Z_1) + g(Y_1, V)\eta(Z_1) + g(Z_1, V)\eta(Y_1) \\ + 2(n-1)g(Y_1, Z_1) = 0.$$

So, we have

Theorem 4.1: *Ricci soliton on para Kenmotsu manifolds admitting Schouten-Van-Kampen connection is invariant if and only if it fulfils the condition equation (4.4).*

Also, if we take $V = \xi$ in equation (4.1), we obtain

$$(4.5) \quad (\mathcal{L}_\xi^*g)(Y_1, Z_1) + 2S^*(Y_1, Z_1) + 2\lambda g(Y_1, Z_1) = 0.$$

Again, we take $V = \xi$ in equation (4.2), we obtain

$$(4.6) \quad (\mathcal{L}_\xi^* g)(Y_1, Z_1) = g(\nabla_{Y_1}^* \xi, Z_1) + g(Y_1, \nabla_{Z_1}^* \xi),$$

from equation (3.2), we get

$$(4.7) \quad (\mathcal{L}_\xi^* g)(Y_1, Z_1) = g[(\nabla_{Y_1} \xi - Y_1 + \eta(Y_1)\xi), Z_1] \\ + g[Y_1, (\nabla_{Z_1} \xi - Z_1 + \eta(Z_1)\xi)].$$

By equations (2.4) and (4.7), we get

$$(4.8) \quad (\mathcal{L}_\xi^* g)(Y_1, Z_1) = 0,$$

using the equations (4.5) and (4.8), we get,

$$(4.9) \quad S^*(Y_1, Z_1) = -\lambda g(Y_1, Z_1),$$

and from the equation (3.7), we obtain

$$(4.10) \quad S(Y_1, Z_1) = -(n - 1 + \lambda)g(Y_1, Z_1).$$

Thus, we have the following result:

Theorem 4.2: *A Ricci soliton on para- Kenmotsu manifolds with respect to Schouten-Van-Kampen connection is an Einstein manifold.*

Next, we take $Z_1 = \xi$ in equation (4.9), we obtain

$$(4.11) \quad S^*(Y_1, \xi) = -\lambda \eta(Y_1).$$

Using the equation (3.7), we have

$$(4.12) \quad \lambda = 0.$$

Therefore, we conclude that

Theorem 4.3: *On Schouten-Van-Kampen connection of para-Kenmotsu manifolds admits Ricci solitons (g, ξ, λ) is steady.*

5. Some curvature results of Ricci solitons on para-Kenmotsu manifolds admitting Schouten-Van-Kampen connection

Concircular curvature on Schouten-Van-Kampen connection of para Kenmotsu manifolds is defined by

$$(5.1) \quad C^*(X_1, Y_1)Z_1 = R^*(X_1, Y_1)Z_1 - \frac{r^*}{n(n-1)}[g(Y_1, Z_1)X_1 - g(X_1, Z_1)Y_1].$$

From equations (3.5), (3.8) and (5.1), we get

$$(5.2) \quad C^*(X_1, Y_1)Z_1 = R(X_1, Y_1)Z_1 - \frac{r}{n(n-1)}[g(Y_1, Z_1)X_1 - g(X_1, Z_1)Y_1],$$

where C^* is the concircular curvature tensor admitting on Schouten-Van-Kampen connection of para- Kenmotsu manifolds. Again, we take $Z_1 = \xi$ in equation (5.2), we get

$$(5.3) \quad C^*(X_1, Y_1)\xi = \frac{r+n(n-1)}{n(n-1)} [\eta(X_1)Y_1 - \eta(Y_1)X_1].$$

Therefore, we conclude that

Theorem 5.1: *A concircular curvature on Schouten-Van-Kampen connection of para Kenmotsu manifolds is determined through equation (5.3).*

Next, we take $Y_1=Z_1=v_i$ in equation (4.10) and contracting it, then we find

$$(5.4) \quad r = -n(n-1+\lambda).$$

Using equations (5.3) and (5.4), we obtain

$$(5.5) \quad C^*(X_1, Y_1)\xi = -\frac{\lambda}{n-1} [\eta(X_1)Y_1 - \eta(Y_1)X_1].$$

From above, we have

Theorem 5.2: *On Schouten-Van-Kampen connection of para-Kenmotsu manifolds admits Ricci solitons is ξ -concircularly flat if and only if $\lambda = 0$.*

Also, the projective curvature tensor is defined on Schouten-Van-Kampen connection of para-Kenmotsu manifolds

$$(5.6) \quad P^*(X_1, Y_1)Z_1 = R^*(X_1, Y_1)Z_1 - \frac{1}{n-1} [S^*(Y_1, Z_1)X_1 - S^*(X_1, Z_1)Y_1],$$

putting $Z_1=\xi$ in equation (5.6), we get

$$(5.7) \quad P^*(X_1, Y_1)\xi = R^*(X_1, Y_1)\xi - \frac{1}{n-1} [S^*(Y_1, \xi)X_1 - S^*(X_1, \xi)Y_1],$$

using equations (3.9) and (4.9) in equation (5.6), we get

$$(5.8) \quad P^*(X_1, Y_1)\xi = \frac{\lambda}{n-1} [\eta(Y_1)X_1 - \eta(X_1)Y_1].$$

Hence, we conclude that

Theorem 5.3: *A Ricci soliton on para Kenmotsu manifolds admitting Schouten-Van-Kampen connection is ξ -projectively flat if and only if $\lambda = 0$.*

6. Ricci solitons on submanifolds of para-Kenmotsu manifolds admitting Levi-Civita connection

The study of submanifolds of an almost contact manifold is one of the utmost enthralling subject in Differential geometry. According to the behaviour of the tangent bundle of a submanifold with respect to the action of an almost contact structure ϕ of the ambient manifold, there are two well-defined types of submanifolds, firstly invariant submanifolds and another anti-invariant submanifolds. Let m -dimensional $\tilde{M}^m$ be a submanifold of para Kenmotsu manifolds M^n , where $m < n$, with induced metric g . Also, let $\tilde{\nabla}$ and $\tilde{\nabla}^\perp$ be the Levi-Civita connections on the tangent bundle $T\tilde{M}^m$ and the normal

bundle $T\tilde{M}^m$ of $\tilde{M}^m$, respectively. Then the Gauss and Weingarten formulae are represented by

$$(6.1) \quad \nabla_{X_1} Y_1 = \tilde{\nabla}_{X_1} Y_1 + \tilde{\sigma}(X_1, Y_1),$$

$$(6.2) \quad \nabla_{X_1} N = -A_N X_1 + \nabla_{X_1} N,$$

for all X_1, Y_1 in $T\tilde{M}^m$ and N in $T\tilde{M}^m$, where $\tilde{\sigma}$ and A_N are the second fundamental form and the shape operator, respectively for the immersion of $\tilde{M}^m$ into M^n the second fundamental form $\tilde{\sigma}$ and the shape operator A_N are related by

$$(6.3) \quad g(\tilde{\sigma}(X_1, Y_1), N) = g(A_N X_1, Y_1),$$

for any X_1, Y_1 in $T\tilde{M}^m$. A submanifold $\tilde{M}^m$ of an almost contact metric manifold M^n is said to be invariant if the structure vector field ξ is tangent to $\tilde{M}^m$ for every vector field X_1 tangent to $\tilde{M}^m$ at every point of $\tilde{M}^m$.

Let us take Ricci solitons (g, ξ, λ) on $\tilde{M}^m$ as

$$(6.4) \quad (\tilde{\mathcal{L}}_\xi, g)(Y_1, Z_1) + 2\tilde{S}(Y_1, Z_1) + 2\lambda g(Y_1, Z_1) = 0.$$

Put $Y_1 = \xi$ in equation (6.1) and by equation (2.4), we get

$$(6.5) \quad X_1 - \eta(X_1)\xi = \tilde{\nabla}_{X_1} \xi + \tilde{\sigma}(X_1, \xi).$$

Since $\tilde{M}^m$ is invariant, therefore ξ in $T\tilde{M}^m$. Then, on comparing tangential and normal components of equation (6.5), we obtain

$$(6.6) \quad \tilde{\nabla}_{X_1} \xi = X_1 - \eta(X_1)\xi \quad \text{and} \quad \tilde{\sigma}(X_1, \xi) = 0.$$

By the equation (6.6), we have

$$(6.7) \quad (\tilde{\mathcal{L}}_\xi g)(Y_1, Z_1) = 2(g(Y_1, Z_1) - \eta(Y_1)\eta(Z_1)),$$

by equations (6.1) and (6.4), we get

$$(6.8) \quad \tilde{S}(Y_1, Z_1) = -(\lambda + 1)g(Y_1, Z_1) + \eta(Y_1)\eta(Z_1),$$

again, using equation (6.6) and the curvature formula becomes

$$(6.9) \quad \tilde{R}(X_1, Y_1)\xi = \tilde{\nabla}_{X_1} \tilde{\nabla}_{Y_1} \xi - \tilde{\nabla}_{Y_1} \tilde{\nabla}_{X_1} \xi - \tilde{\nabla}_{[X_1, Y_1]} \xi,$$

$$(6.10) \quad \tilde{R}(X_1, Y_1)\xi = \eta(X_1)Y_1 - \eta(Y_1)X_1.$$

On contracting the equation (6.10), we get

$$(6.11) \quad \tilde{S}(X_1, \xi) = (m - 1)\eta(X_1).$$

Put $Z_1 = \xi$ in equation (6.8) and using equation (6.11), we obtain

$$(6.12) \quad \lambda = -(m - 1).$$

By equations (6.8) and (6.12), we obtain the following results:

Theorem 6.1 Ricci solitons (g, ξ, λ) on invariant submanifolds $\tilde{M}^m$ of para-Kenmotsu manifolds M^n then $\tilde{M}^m$ is:

1. η -Einstein manifold.
2. Always shrinking.

7. Ricci solitons on submanifolds of para-Kenmotsu manifolds equipped with the Schouten-Van-Kampen connection

Suppose that Levi-Civita connection $\tilde{\nabla}$ and Schouten-Van-Kampen connection $\tilde{\nabla}^*$ on submanifolds $\tilde{M}^m$ of para Kenmotsu manifolds M^n . Here we denote the second fundamental form with respect to Schouten-Van-Kampen connection $\tilde{\sigma}$, then the Gauss and Weingarten formulae are given by

$$(7.1) \quad \nabla_{X_1}^* Y_1 = \tilde{\nabla}_{X_1}^* Y_1 + \tilde{\sigma}^*(X_1, Y_1),$$

Again, by equation (3.2), we obtain

$$(7.2) \quad \tilde{\nabla}_{X_1}^* Y_1 + \tilde{\sigma}^*(X_1, Y_1) = \nabla_{X_1} Y_1 - \eta(Y_1)X_1 + g(X_1, Y_1)\xi.$$

Taking equations (6.1) and (7.2), we get

$$(7.3) \quad \tilde{\nabla}_{X_1}^* Y_1 + \tilde{\sigma}^*(X_1, Y_1) = \tilde{\nabla}_{X_1} Y_1 + \tilde{\sigma}(X_1, Y_1) - \eta(Y_1)X_1 + g(X_1, Y_1)\xi.$$

Since $\tilde{M}^m$ is an invariant submanifold, the tangential and normal parts are given by

$$(7.4) \quad \tilde{\nabla}_{X_1}^* Y_1 = \tilde{\nabla}_{X_1} Y_1 - \eta(Y_1)X_1 + g(X_1, Y_1)\xi,$$

and

$$(7.5) \quad \tilde{\sigma}^*(X_1, Y_1) = \tilde{\sigma}(X_1, Y_1).$$

So, we state the following:

Theorem 7.1: If $\tilde{M}^m$ is an invariant submanifold of para-Kenmotsu manifolds M^n endowed with two connections, namely Levi-Civita connection and Schouten-Van-Kampen connection ∇, ∇^* and $\tilde{\nabla}, \tilde{\nabla}^*$ be the induced Levi-Civita and Schouten-Van-Kampen connection on $\tilde{M}^m$ from ∇, ∇^* respectively. Then we have

1. $\tilde{M}^m$ admits Schouten-Van-Kampen connection.
2. The second fundamental form are same with respect to ∇ and ∇^* .

Next, the curvature tensor, Ricci tensor and scalar curvature tensor on $\tilde{M}^m$ of M^n are given as

Theorem 7.2: The curvature tensor $\tilde{R}^*$ is given as

$$(7.6) \quad \tilde{R}^*(X_1, Y_1)Z_1 = \tilde{R}(X_1, Y_1)Z_1 - g(X_1, Z_1)Y_1 + g(Y_1, Z_1)X_1.$$

Ricci tensor $\tilde{S}^*$ is represented as

$$(7.7) \quad \tilde{S}^*(Y_1, Z_1) = \tilde{S}(Y_1, Z_1) + (n - 1)g(Y_1, Z_1).$$

The Scalar curvature is given by

$$(7.8) \quad \tilde{r}^* = \tilde{r} + n(n-1).$$

Also, if (g, ξ, λ) on an invariant submanifold $\tilde{M}^m$ of para-Kenmotsu manifolds M^n equipped with Shouten-Van-Kampen connection, which is defined as

$$(7.9) \quad (\tilde{\mathcal{L}}_\xi^* g)(Y_1, Z_1) + 2\tilde{S}^*(Y_1, Z_1) + 2\lambda g(Y_1, Z_1) = 0.$$

By adopting similar step done in proving theorems 4.2 and 4.3, we obtain the following results:

Theorem 7.3: *If (g, ξ, λ) be a Ricci soliton on an invariant submanifold $\tilde{M}^m$ of para-Kenmotsu manifolds admitting the Schouten-Van-Kampen connection, then $\tilde{M}^m$ is:*

1. Einstein manifolds.
2. Always steady.

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