

On Certain New Tensors in an Areal Space Based on Parameter Invariant Multiple Integrals

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Abstract: Areal spaces of different types have been studied by various eminent Mathematicians including Cartan¹, Davies^{2,3}, Kawaguchi⁴ and others. The study of Areal spaces through parameter invariant multiple integrals was initiated by Rund^{5,6}. Rund made a significant change in the study of Areal spaces by introducing four index metric- tensor in place of usual two index metric tensor. Various aspects of this study have been continued by several authors including Lovelock and Rund⁷, Rastogi⁸⁻¹² etc. The purpose of the present paper is to introduce certain new tensors in Areal space and study their properties vice-versa other well- known tensors of Areal spaces.

Keywords: Areal space, Tensor D_{ijk}^{ABE} , Tensor E_{ijk}^{ABD} , areal space X_4 , the subspace C_2 .

1. Introduction

Let X_n be an n -dimensional differentiable manifold with local coordinates x^i ($i = 1, \dots, n$) and C_r be an r -dimensional subspace represented by $x^i = x^i(t^A)$, $A = 1, \dots, r$, ($r < n$), where t^A denotes a system of r -independent parameters on C_r such that Rund⁶ $x'^h_A = \partial x^h / \partial t^A$. In such a space let $L(x^h, x'^h_A)$ be a function of $n + nr$ variables x^h and x'^h_A , which satisfies Rund¹²

$$(1.1) \quad (\partial'^i_A L) x'^i_B = L \delta^A_B.$$

Let $g_{ij}^{AB}(x^h, x'^h_E)$ be the metric tensor of X_n , then we have

$$(1.2) \quad g_{ij}^{AB}(x^h, x'^h_E) = \frac{1}{2}r(\partial'_{Bj}\partial'_{Ai}L^{2/r}),$$

which has the reciprocal tensor $g_{AB}^{ij}(x^h, x'^h_E)$, such that

$$(1.3) \quad g_{ik}^{AC} g_{AB}^{ij} = \delta_k^j \delta_C^B.$$

Corresponding to the metric tensor, we have the torsion tensor C_{ijk}^{ABD} satisfying

$$(1.4) \quad C_{ijk}^{ABD} = \frac{1}{2}(\partial_k'^D g_{ij}^{AB}).$$

For a set of linearly independent differentiable vector field X_A^i in X_n tangent to C_r , we have the following covariant derivative of X_A^i

$$(1.5) \quad X_{A/j}^i = \partial_j X_A^i - (\partial_k'^D X_A^i) \Gamma_{mj}^k x_D'^m + \Gamma_{jm}^i X_A^m,$$

where all the terms have their usual meaning Rund^{11, 12}.

In the next section we shall study properties of the space X_4 parameterized by C_2 .

2. Space X_4 Parameterized by C_2

In case of an areal space of four dimensions parametrised by C_2 , let us suppose that we have the vector

$$(2.1a) \quad L_i^A = \frac{1}{2}L^{-1/2}(\partial_i'^A L),$$

and

$$(2.1b) \quad L_i^A = L^{-1/2}x_A'^i,$$

such that

$$(2.1) \quad L_i^A L_A^i = 1, \quad L_i^1 x_1'^i = \frac{1}{2}L^{1/2}, \quad L_i^1 x_2'^i = 0,$$

$$(2.1c) \quad L_i^2 x_1'^i = 0, \quad L_i^2 x_2'^i = \frac{1}{2}L^{1/2}.$$

We can alternatively also write

$$(2.2) \quad L_i^A x_B'^i = \frac{1}{2}L^{1/2}\delta_B^A.$$

If we define

$$(2.3) \quad L_{ij}^{AB} = \partial_j'^B L_i^A,$$

we can obtain

$$(2.4) \quad L_{ij}^{AB} x_D'^i = \frac{1}{2}(\delta_D^A L_j^B - \delta_D^B L_j^A),$$

which leads to

$$(2.5) \quad L_{ij}^{AB} x_A'^i = 0, \quad L_{ij}^{AB} x_B'^i = -\frac{3}{2}L_j^A.$$

Also, the metric tensor can be defined as

$$(2.6) \quad g_{ij}^{AB} = 2(L^{1/2}L_{ij}^{AB} + L_i^A L_j^B).$$

If we define

$$(2.7) \quad h_{ij}^{AB} = 2L^{1/2}L_{ij}^{AB} = (g_{ij}^{AB} - 2L_i^A L_j^B),$$

we can obtain

$$(2.8a) \quad h_{ij}^{AB} x'^i_D = L^{1/2}(\delta_D^A L_j^B - 2\delta_D^B L_j^A), \quad h_{ij}^{AB} x'^i_A = 0,$$

$$(2.8b) \quad h_{ij}^{AB} x'^i_B = -3L^{1/2}L_j^A, \quad h_{ij}^{AB} x'^i_B x'^j_A = -3L,$$

and

$$(2.8c) \quad C_{ijk}^{ABE} x'^i_D = (\delta_D^E g_{kj}^{AB} + \delta_D^B g_{jk}^E - \delta_D^A g_{jk}^{BE}).$$

From these equations we can easily obtain the following theorem.

Theorem 2.1. *In an areal space X_4 , parametrised by C_2 , vanishing of h_{ij}^{AB} is a sufficient condition for C_{ijk}^{ABD} to vanish. Conversely, if $C_{ijk}^{ABD} = 0$, the tensor h_{ij}^{AB} will be homogeneous of degree zero in x'^i_D .*

3. Tensor D_{ijk}^{ABE}

In an areal space X_4 parametrised by C_2 , we have the following mutually orthogonal unit vectors namely M_A^i , $N_{(1)A}^i$ and $N_{(2)A}^i$ such that they satisfy

$$(3.1) \quad \begin{aligned} L_i^A M_A^i &= 0, & M_i^A M_A^i &= 1, & L_i^A N_{(1)A}^i &= 0, & M_i^A N_{(1)A}^i &= 0, \\ N_{(1)i}^A N_{(1)A}^i &= 1, & L_i^A N_{(2)A}^i &= 0, & M_i^A N_{(2)A}^i &= 0, \\ N_{(1)i}^A N_{(2)A}^i &= 0, & N_{(2)i}^A N_{(2)A}^i &= 1. \end{aligned}$$

The tensor C_{ijk}^{ABD} in the space X_4 parametrised by C_2 , explicitly in terms of unit vectors, is expressed as

$$(3.2) \quad \begin{aligned} C_{ijk}^{ABD} &= C_{(1)} M_i^A M_j^B M_k^D + C_{(2)} N_{(1)i}^A N_{(1)j}^B N_{(1)k}^C + C_{(3)} N_{(2)i}^A N_{(2)j}^B N_{(2)k}^C \\ &+ \sum_{(ijk)}^{ABD} \{ C_{(4)} M_i^A M_j^B N_{(1)k}^D + C_{(5)} M_i^A M_j^B N_{(2)k}^D + C_{(6)} N_{(1)i}^A N_{(1)j}^B M_k^D \\ &+ C_{(7)} N_{(1)i}^A N_{(1)j}^B N_{(2)k}^D + C_{(8)} N_{(2)i}^A N_{(2)j}^B M_k^D + C_{(9)} N_{(2)i}^A N_{(2)j}^B N_{(1)k}^D \\ &+ C_{(10)} M_i^A (N_{(1)j}^B N_{(2)k}^D + N_{(2)j}^B N_{(1)k}^D) \}. \end{aligned}$$

Using these vectors, we define a new tensor D_{ijk}^{ABE} similar to C_{ijk}^{ABE} , which satisfies

$$(3.3) \quad D_{ijk}^{ABE} L_A^i = 0, \quad D_{ijk}^{ABE} g_{AB}^{ij} = D_k^E,$$

where $D_k^E = DN_{(1)k}^E$. Any such tensor can be expressed as

$$(3.4) \quad \begin{aligned} D_{ijk}^{ABE} = & D_{(1)}M_i^A M_j^B M_k^E + D_{(2)}N_{(1)i}^A N_{(1)j}^B N_{(1)k}^E + D_{(3)}N_{(2)i}^A N_{(2)j}^B N_{(2)k}^E \\ & + \sum_{(ijk)}^{ABE} \{M_i^A M_j^B (D_{(4)}N_{(1)k}^E + D_{(5)}N_{(2)k}^E) + N_{(1)i}^A N_{(1)j}^B (D_{(6)}M_k^E \\ & + D_{(7)}N_{(2)k}^E) + N_{(2)i}^A N_{(2)j}^B (D_{(8)}M_k^E + D_{(9)}N_{(1)k}^E) \\ & + D_{(10)}M_i^A (N_{(1)j}^B N_{(2)k}^E + N_{(2)j}^B N_{(1)k}^E)\}. \end{aligned}$$

Multiplying equation (3.4) by g_{BE}^{jk} and simplifying, we obtain

$$(3.5) \quad \begin{aligned} D_i^A = & (D_{(1)} + D_{(6)} + D_{(8)})M_i^A + (D_{(2)} + D_{(4)} + D_{(9)})N_{(1)i}^A \\ & + (D_{(3)} + D_{(5)} + D_{(7)})N_{(2)i}^A. \end{aligned}$$

Comparing equation (3.5) with $D_k^E = DN_{(1)k}^E$, we obtain

$$(3.6) \quad \begin{aligned} D_{(2)} + D_{(4)} + D_{(9)} = D, \quad D_{(1)} + D_{(6)} + D_{(8)} = 0, \\ D_{(3)} + D_{(5)} + D_{(7)} = 0. \end{aligned}$$

Theorem 3.1. *In a four-dimensional areal space X_4 parametrised by a two-dimensional subspace C_2 , the tensor D_{ijk}^{ABE} is symmetric in the pairs of indices (A, i) , (B, j) and (E, k) and is expressed by equation (3.4), with coefficients satisfying equation (3.6).*

Multiplying equation (3.4) by M_A^i , $N_{(1)A}^i$ and $N_{(2)A}^i$ respectively and using

$$(3.7) \quad D_{ijk}^{ABE} M_A^i = {}^1 D_{jk}^{BE}, \quad D_{ijk}^{ABE} N_{(1)A}^i = {}^2 D_{jk}^{BE}, \quad D_{ijk}^{ABE} N_{(2)A}^i = {}^3 D_{jk}^{BE},$$

we obtain,

$$(3.8a) \quad \begin{aligned} {}^1 D_{jk}^{BE} = & M_j^B (D_{(1)}M_k^E + D_{(4)}N_{(1)k}^E + D_{(5)}N_{(2)k}^E) \\ & + N_{(1)j}^B (D_{(4)}M_k^E + D_{(6)}N_{(1)k}^E + D_{(10)}N_{(2)k}^E) \\ & + N_{(2)j}^B (D_{(5)}M_k^E + D_{(10)}N_{(1)k}^E + D_{(8)}N_{(2)k}^E), \end{aligned}$$

$$(3.8b) \quad \begin{aligned} {}^2 D_{jk}^{BE} = & M_j^B (D_{(4)}M_k^E + D_{(6)}N_{(1)k}^E + D_{(10)}N_{(2)k}^E) \\ & + N_{(1)j}^B (D_{(6)}M_k^E + D_{(2)}N_{(1)k}^E + D_{(7)}N_{(2)k}^E) \\ & + N_{(2)j}^B (D_{(10)}M_k^E + D_{(7)}N_{(1)k}^E + D_{(9)}N_{(2)k}^E), \end{aligned}$$

$$(3.8c) \quad \begin{aligned} {}^3 D_{jk}^{BE} = & M_j^B (D_{(5)}M_k^E + D_{(10)}N_{(1)k}^E + D_{(8)}N_{(2)k}^E) \\ & + N_{(1)j}^B (D_{(10)}M_k^E + D_{(7)}N_{(1)k}^E + D_{(9)}N_{(2)k}^E) \\ & + N_{(2)j}^B (D_{(8)}M_k^E + D_{(9)}N_{(1)k}^E + D_{(3)}N_{(2)k}^E). \end{aligned}$$

Using equations (3.4), (3.7) and (3.8a), (3.8b) and (3.8b), we get

$$(3.9) \quad D_{ijk}^{ABE} = {}^1D_{jk}^{BE} M_i^A + {}^2D_{jk}^{BE} N_{(1)i}^A + {}^3D_{jk}^{BE} N_{(2)i}^A.$$

Theorem 3.2. *In an areal space X_4 parametrised by the subspace C_2 , the tensor D_{ijk}^{ABE} can be decomposed in the form of equation (3.9).*

From equations (3.8a), (3.8b) and (3.8c), we obtain

$$(3.10a) \quad {}^1D_{jk}^{BE} M_B^j = D_{(1)} M_k^E + D_{(4)} N_{(1)k}^E + D_{(5)} N_{(2)k}^E,$$

$$(3.10b) \quad {}^1D_{jk}^{BE} N_{(1)B}^j = D_{(4)} M_k^E + D_{(6)} N_{(1)k}^E + D_{(10)} N_{(2)k}^E = {}^2D_{jk}^{BE} M_B^j,$$

$$(3.10c) \quad {}^1D_{jk}^{BE} N_{(2)B}^j = D_{(5)} M_k^E + D_{(10)} N_{(1)k}^E + D_{(8)} N_{(2)k}^E = {}^3D_{jk}^{BE} M_B^j,$$

$$(3.10d) \quad {}^2D_{jk}^{BE} N_{(1)B}^j = D_{(6)} M_k^E + D_{(2)} N_{(1)k}^E + D_{(7)} N_{(2)k}^E,$$

$$(3.10e) \quad {}^2D_{jk}^{BE} N_{(2)B}^j = D_{(10)} M_k^E + D_{(7)} N_{(1)k}^E + D_{(9)} N_{(2)k}^E = {}^3D_{jk}^{BE} N_{(1)B}^j,$$

$$(3.10f) \quad {}^3D_{jk}^{BE} N_{(2)B}^j = D_{(8)} M_k^E + D_{(9)} N_{(1)k}^E + D_{(3)} N_{(2)k}^E.$$

Hence, the coefficients $D_{(1)}$ to $D_{(10)}$ are given by

$$(3.11a) \quad D_{(1)} = {}^1D_{jk}^{BE} M_B^j M_E^k, \quad D_{(2)} = {}^2D_{jk}^{BE} N_{(1)B}^j N_{(1)E}^k,$$

$$(3.11b) \quad D_{(3)} = {}^3D_{jk}^{BE} N_{(2)B}^j N_{(2)E}^k, \quad D_{(4)} = {}^2D_{jk}^{BE} M_B^j M_E^k,$$

$$(3.11c) \quad D_{(5)} = {}^3D_{jk}^{BE} M_B^j M_E^k, \quad D_{(6)} = {}^1D_{jk}^{BE} N_{(1)B}^j N_{(1)E}^k,$$

$$(3.11d) \quad D_{(7)} = {}^2D_{jk}^{BE} N_{(1)B}^j N_{(2)E}^k, \quad D_{(8)} = {}^3D_{jk}^{BE} N_{(2)B}^j M_E^k,$$

$$(3.11e) \quad D_{(9)} = {}^3D_{jk}^{BE} N_{(2)B}^j N_{(1)E}^k, \quad D_{(10)} = {}^1D_{jk}^{BE} N_{(1)B}^j N_{(2)E}^k.$$

Theorem 3.3. *In an areal space X_4 parametrised by C_2 , the coefficients $D_{(1)}$ to $D_{(10)}$ are given by equations (3.11 a)–(3.11 e).*

From equations (3.8a), (3.8b) and (3.8c), we also obtain

$$(3.12a) \quad {}^1D_{jk}^{BE} M_B^j = D_{(1)} M_k^E + D_{(4)} N_{(1)k}^E + D_{(6)} N_{(2)k}^E,$$

$$(3.12b) \quad {}^2D_{jk}^{BE} N_{(1)B}^j = D_{(6)} M_k^E + D_{(2)} N_{(1)k}^E + D_{(7)} N_{(2)k}^E,$$

$$(3.12c) \quad {}^3D_{jk}^{BE} N_{(2)B}^j = D_{(8)} M_k^E + D_{(9)} N_{(1)k}^E + D_{(3)} N_{(2)k}^E.$$

Adding these equations and using equation (3.5), we obtain

$$(3.13) \quad {}^1D_{jk}^{BE} M_B^j + {}^2D_{jk}^{BE} N_{(1)B}^j + {}^3D_{jk}^{BE} N_{(2)B}^j = D_k^E.$$

Theorem 3.4. *An areal space X_4 parametrised by C_2 satisfies relation (3.13).*

4. Tensor E_{ijk}^{ABD}

Similar to the tensors C_{ijk}^{ABD} and D_{ijk}^{ABE} defined earlier, we define a new tensor E_{ijk}^{ABD} such that it satisfies $E_{ijk}^{ABD} L_A^i = 0$ and $E_{ijk}^{ABD} g_{AB}^{ij} = E_k^D$, where $E_k^D = E N_{(2)k}^D$, in the following form

$$(4.1) \quad \begin{aligned} E_{ijk}^{ABD} = & E_{(1)} M_i^A M_j^B M_k^D + E_{(2)} N_{(1)i}^A N_{(1)j}^B N_{(1)k}^D + E_{(3)} N_{(2)i}^A N_{(2)j}^B N_{(2)k}^D \\ & + \sum_{(ijk)}^{ABD} \{ E_{(4)} M_i^A M_j^B N_{(1)k}^D + E_{(5)} M_i^A M_j^B N_{(2)k}^D + E_{(6)} N_{(1)i}^A N_{(1)j}^B M_k^D \\ & + E_{(7)} N_{(1)i}^A N_{(1)j}^B N_{(2)k}^D + E_{(8)} N_{(2)i}^A N_{(2)j}^B M_k^D + E_{(9)} N_{(2)i}^A N_{(2)j}^B N_{(1)k}^D \\ & + E_{(10)} M_i^A (N_{(1)j}^B N_{(2)k}^D + N_{(2)j}^B N_{(1)k}^D) \}. \end{aligned}$$

Multiplying equation (4.1) by g_{AB}^{ij} and simplifying, we obtain

$$(4.2) \quad \begin{aligned} E_k^D = & (E_{(1)} + E_{(6)} + E_{(8)}) M_k^D + (E_{(2)} + E_{(4)} + E_{(9)}) N_{(1)k}^D \\ & + (E_{(3)} + E_{(5)} + E_{(7)}) N_{(2)k}^D. \end{aligned}$$

Comparing equation (4.2) with $E_{Dk} = E N_{Dk}^{(2)}$, we obtain

$$(4.3) \quad \begin{aligned} E = E_{(3)} + E_{(5)} + E_{(7)}, \quad E_{(1)} + E_{(6)} + E_{(8)} = 0, \\ E_{(2)} + E_{(4)} + E_{(9)} = 0. \end{aligned}$$

Theorem 4.1. *In a four-dimensional areal space X_4 parametrised by a two-dimensional subspace C_2 , the tensor E_{ijk}^{ABD} is symmetric in the pairs of indices (A, i) , (B, j) and (D, k) and is expressed by equation (4.1), with coefficients satisfying equation (4.3).*

Multiplying equation (4.1) by M_A^i , $N_{(1)A}^i$ and $N_{(2)A}^i$ respectively and using

$$(4.4) \quad E_{ijk}^{ABD} M_A^i = {}^1 E_{jk}^{BD}, \quad E_{ijk}^{ABD} N_{(1)A}^i = {}^2 E_{jk}^{BD}, \quad E_{ijk}^{ABD} N_{(2)A}^i = {}^3 E_{jk}^{BD},$$

we obtain

$$(4.5a) \quad \begin{aligned} {}^1 E_{jk}^{BD} = & M_j^B (E_{(1)} M_k^D + E_{(4)} N_{(1)k}^D + E_{(5)} N_{(2)k}^D) \\ & + N_{(1)j}^B (E_{(4)} M_k^D + E_{(6)} N_{(1)k}^D + E_{(10)} N_{(2)k}^D) \\ & + N_{(2)j}^B (E_{(5)} M_k^D + E_{(10)} N_{(1)k}^D + E_{(8)} N_{(2)k}^D), \end{aligned}$$

$$(4.5b) \quad \begin{aligned} {}^2 E_{jk}^{BD} = & M_j^B (E_{(4)} M_k^D + E_{(6)} N_{(1)k}^D + E_{(10)} N_{(2)k}^D) \\ & + N_{(1)j}^B (E_{(6)} M_k^D + E_{(2)} N_{(1)k}^D + E_{(7)} N_{(2)k}^D) \\ & + N_{(2)j}^B (E_{(10)} M_k^D + E_{(7)} N_{(1)k}^D + E_{(9)} N_{(2)k}^D), \end{aligned}$$

$$\begin{aligned}
 {}^3E_{jk}^{BD} = & M_j^B (E_{(5)}M_k^D + E_{(10)}N_{(1)k}^D + E_{(8)}N_{(2)k}^D) \\
 (4.5c) \quad & + N_{(1)j}^B (E_{(10)}M_k^D + E_{(7)}N_{(1)k}^D + E_{(9)}N_{(2)k}^D) \\
 & + N_{(2)j}^B (E_{(8)}M_k^D + E_{(9)}N_{(1)k}^D + E_{(3)}N_{(2)k}^D).
 \end{aligned}$$

Using equations (4.1), (4.2) and (4.5 a)–(4.5 c), we obtain

$$(4.6) \quad E_{ijk}^{ABD} = {}^1E_{jk}^{BD}M_i^A + {}^2E_{jk}^{BD}N_{(1)i}^A + {}^3E_{jk}^{BD}N_{(2)i}^A.$$

Theorem 4.2. *In an areal space X_4 parametrised by the subspace C_2 , the tensor E_{ijk}^{ABE} can be decomposed in the form of equation (4.6).*

From equations (4.5a), (4.5b) and (4.5c), we obtain

$$(4.7a) \quad {}^1E_{jk}^{BD}M_B^j = E_{(1)}M_k^D + E_{(4)}N_{(1)k}^D + E_{(5)}N_{(2)k}^D,$$

$$(4.7b) \quad {}^1E_{jk}^{BD}N_{(1)B}^j = E_{(4)}M_k^D + E_{(6)}N_{(1)k}^D + E_{(10)}N_{(2)k}^D = {}^2E_{jk}^{BD}M_B^j,$$

$$(4.7c) \quad {}^1E_{jk}^{BD}N_{(2)B}^j = E_{(5)}M_k^D + E_{(10)}N_{(1)k}^D + E_{(8)}N_{(2)k}^D = {}^3E_{jk}^{BD}M_B^j,$$

$$(4.7d) \quad {}^2E_{jk}^{BD}N_{(1)B}^j = E_{(6)}M_k^D + E_{(2)}N_{(1)k}^D + E_{(7)}N_{(2)k}^D,$$

$$(4.7e) \quad {}^2E_{jk}^{BD}N_{(1)B}^j = E_{(10)}M_k^D + E_{(7)}N_{(1)k}^D + E_{(9)}N_{(2)k}^D = {}^3E_{jk}^{BD}N_{(1)B}^j,$$

$$(4.7f) \quad {}^3E_{jk}^{BD}N_{(2)B}^j = E_{(8)}M_k^D + E_{(9)}N_{(1)k}^D + E_{(3)}N_{(2)k}^D.$$

Hence, the coefficients $E_{(1)}$ to $E_{(10)}$ are given by

$$(4.8a) \quad E_{(1)} = {}^1E_{jk}^{BD}M_B^jM_D^k, \quad E_{(2)} = {}^2E_{jk}^{BD}N_{(1)B}^jN_{(1)D}^k,$$

$$(4.8b) \quad E_{(3)} = {}^3E_{jk}^{BD}N_{(2)B}^jN_{(2)D}^k, \quad E_{(4)} = {}^2E_{jk}^{BD}M_B^jM_D^k,$$

$$(4.8c) \quad E_{(5)} = {}^3E_{jk}^{BD}M_B^jM_D^k, \quad E_{(6)} = {}^1E_{jk}^{BD}N_{(1)B}^jN_{(1)D}^k,$$

$$(4.8d) \quad E_{(7)} = {}^3E_{jk}^{BD}N_{(1)B}^jN_{(1)D}^k, \quad E_{(8)} = {}^1E_{jk}^{BD}N_{(2)B}^jN_{(2)D}^k,$$

$$(4.8e) \quad E_{(9)} = {}^2E_{jk}^{BD}N_{(2)B}^jN_{(2)D}^k, \quad E_{(10)} = {}^2E_{jk}^{BD}M_B^jM_D^k.$$

Hence:

Theorem 4.3. *In an areal space X_4 parametrised by the subspace C_2 , the coefficients $E_{(1)}$ to $E_{(10)}$ are given by equations (4.8)a–(4.8)e.*

Adding equations (4.7a), (4.7d) and (4.7f), we obtain

$$\begin{aligned}
 (4.9) \quad {}^1E_{jk}^{BD}M_B^j + {}^2E_{jk}^{BD}N_{(1)B}^j + {}^3E_{jk}^{BD}N_{(2)B}^j = & (E_{(1)} + E_{(6)} + E_{(8)})M_k^D \\
 & + (E_{(2)} + E_{(4)} + E_{(9)})N_{(1)k}^D + (E_{(3)} + E_{(5)} + E_{(7)})N_{(2)k}^D.
 \end{aligned}$$

Using equations (4.2) and (4.3), this reduces to

$$(4.10) \quad {}^1E_{jk}^{BD} M_B^j + {}^2E_{jk}^{BD} N_{(1)B}^j + {}^3E_{jk}^{BD} N_{(2)B}^j = E_k^D.$$

Theorem 4.4. *An areal space X_4 parametrised by the subspace C_2 satisfies equation (4.10).*

5. Conclusion

In this paper, we have studied four-dimensional areal space X_4 parameterised by a two-dimensional subspace C_2 in the framework of parameter-invariant multiple integrals. Certain new tensors associated with the four-index metric tensor have been introduced and their fundamental properties have been examined. The tensor h_{ij}^{AB} has been defined and its relation with the torsion tensor C_{ijk}^{ABD} has been established. It has been shown that the vanishing of h_{ij}^{AB} implies the vanishing of the torsion tensor, while the converse condition ensures the homogeneity of h_{ij}^{AB} of degree zero in the areal variables.

Furthermore, we have constructed two new tensors D_{ijk}^{ABE} and E_{ijk}^{ABD} and decomposed explicitly in terms of mutually orthogonal unit vectors. The necessary conditions on their scalar coefficients and their symmetry properties have been derived. These results contribute to the structural understanding of areal spaces and may be useful in the study of variational problems and related geometrical theories.

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