

Concircular Curvature Tensor of Para-Kenmotsu Manifolds admitting Quarter-Symmetric Metric Connection

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Abstract: In this article, we analyze para-Kenmotsu manifolds endowed with a quarter-symmetric metric connection, emphasizing their geometric and curvature characteristics. We investigated quasi-concircularly flat and ϕ -concircularly flat para-Kenmotsu manifolds concerning the quarter-symmetric metric connection. It was established that a ϕ -concircularly flat para-Kenmotsu manifold is an η -Einstein manifold. Additionally, we derived conditions for para-Kenmotsu manifolds satisfying $\tilde{C}(\xi, X_4) \cdot \tilde{S} = 0$ under the influence of the quarter-symmetric metric connection $\tilde{\nabla}$ and proved several significant results.

Keywords: Para-Kenmotsu manifold, Concircular curvature tensor, Quarter-symmetric metric connection.

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1. Introduction

I. Sato¹ introduced the concept of almost paracontact manifolds as the odd-dimensional counterpart of para-complex structures. This idea has attracted significant interest in differential geometry due to its geometric properties and applications in mathematical physics. While almost contact manifolds² are always of odd dimension, almost paracontact manifolds can exist in even dimensions as well. Later, T. Adati and K. Matsumoto³ studied P-Sasakian and SP-Sasakian manifolds, which represent a specific subclass of almost paracontact Riemannian manifolds. Before Sato's work, Takahashi⁴ introduced the concept of almost contact manifolds, including Sasakian

manifolds, in the context of a pseudo-Riemannian metric. In 1985, Kaneyuki and Williams⁵ formalized the notion of almost paracontact structures on pseudo-Riemannian manifolds of dimension $n = 2m + 1$. Subsequently, Zamkovoy⁶ established that every almost paracontact structure can be associated with a pseudo-Riemannian metric of signature $(n + 1, n)$. Welyczko⁷ introduced the para-Kenmotsu manifold as an analogue of the Kenmotsu manifold² within the framework of paracontact geometry. Additionally, significant contributions to the study of para-Kenmotsu manifolds have been made by various researchers⁸⁻¹⁰.

The analysis of Riemannian and semi-Riemannian manifolds equipped with different types of connections is essential for understanding their geometric and algebraic structures. In 1975, Golab¹¹ first introduced the concept of a quarter-symmetric metric connection in differentiable manifolds, extending the idea of semi-symmetric connections.

A linear connection $\tilde{\nabla}$ on a Riemannian manifold (M^n, g) is called a quarter-symmetric metric connection if its torsion tensor $\tilde{T}$ satisfies the relation

$$(1.1) \quad \tilde{T}(X_1, X_2) = \tilde{\nabla}_{X_1} X_2 - \tilde{\nabla}_{X_2} X_1 - [X_1, X_2],$$

which simplifies to

$$(1.2) \quad \tilde{T}(X_1, X_2) = \eta(X_2)\phi X_1 - \eta(X_1)\phi X_2,$$

where η is a 1-form and ϕ is a tensor of type $(1, 1)$. The connection $\tilde{\nabla}$ is a quarter-symmetric metric connection if it satisfies the condition

$$(1.3) \quad (\tilde{\nabla}_{X_1} g)(X_2, X_3) = 0,$$

for all $X_1, X_2, X_3 \in \chi(M)$. When $\phi X_1 = X_1$ and $\phi X_2 = X_2$, the quarter-symmetric connection reduces to a semi-symmetric connection¹². Rastogi further explored the properties of quarter-symmetric metric connections in his studies¹³⁻¹⁴. Additional research on these connections in Hermitian and Kaehlerian manifolds was carried out by Yano and Imai¹⁵. Over the years, numerous researchers, including Biswas and De¹⁶, Sular¹⁷, De and Mondal¹⁸, De and De¹⁹, Singh and Pandey²⁰, and others, have contributed to this area.

On the other hand, concircular curvature tensor is a significant concept in differential geometry, particularly in the study of the geometric structures of Riemannian and pseudo-Riemannian manifolds. It provides a framework for analyzing the concircular geometry of a manifold, which concerns the preservation of concircular circles (geodesic circles) under specific transformations. In 1940, K. Yano²¹ introduced concircular transformation whose geometry is known as concircular geometry.

Concircular curvature tensor of para-Kenmotsu manifold is given by

$$(1.4) \quad C(X_1, X_2)X_3 = R(X_1, X_2)X_3 - \frac{r}{n(n-1)} [g(X_2, X_3)X_1 - g(X_1, X_3)X_2],$$

where $X_1, X_2, X_3 \in \chi(M)$, R and r are the Riemannian curvature tensor and scalar curvature tensors respectively.

The concircular curvature tensor $\tilde{C}$ with respect to quarter-symmetric metric connection $\tilde{\nabla}$ is given by

$$(1.5) \quad \tilde{C}(X_1, X_2)X_3 = \tilde{R}(X_1, X_2)X_3 - \frac{\tilde{r}}{n(n-1)} [g(X_2, X_3)X_1 - g(X_1, X_3)X_2],$$

where $\tilde{R}$, and $\tilde{r}$ are Riemannian curvature tensor and scalar curvature tensor with respect to quarter-symmetric metric connection $\tilde{\nabla}$ respectively. A wide range of studies of the properties and applications of the concircular curvature tensor have focused by various researchers such as²²⁻²³.

An n -dimensional para-Kenmotsu manifold M^n is said to be Einstein manifold if the Ricci tensor S is of the form $S(X_1, X_2) = ag(X_1, X_2)$, where a is scalar.

An n -dimensional para-Kenmotsu manifold M^n is said to be η -Einstein manifold if the Ricci tensor S is of the form $S(X_1, X_2) = ag(X_1, X_2) + b\eta(X_1)\eta(X_2)$, for all $X_1, X_2 \in \chi(M)$, where a and b are scalars.

2. Para-Kenmotsu Manifolds

The concept of an almost para-contact manifold was first introduced by I. Sato¹. An n -dimensional differentiable manifold M^n is said to possess an almost para-contact structure (ϕ, ξ, η) , where ϕ is a tensor field of type $(1,1)$, ξ is a characteristic vector field, and η is a 1-form. These satisfy the following conditions:

$$(2.1) \quad \phi^2 X_1 = X_1 - \eta(X_1)\xi,$$

$$(2.2) \quad \eta \circ \phi = 0,$$

$$(2.3) \quad \phi(\xi) = 0,$$

$$(2.4) \quad \eta(\xi) = 1.$$

A differentiable manifold equipped with the almost para-contact structure (ϕ, ξ, η) is termed an almost para-contact Riemannian manifold. Furthermore, if the manifold M^n admits a semi-Riemannian metric g that satisfies:

$$(2.5) \quad \eta(X_1) = g(X_1, \xi),$$

$$(2.6) \quad g(\phi X_1, \phi X_2) = -g(X_1, X_2) + \eta(X_1)\eta(X_2),$$

then the structure (ϕ, ξ, η, g) , fulfilling conditions (2.2) to (2.6), is known as an almost para-contact Riemannian structure, and the corresponding manifold M^n is referred to as an almost para-contact Riemannian manifold¹.

On a para-Kenmotsu manifold¹, the following geometric relations hold:

$$(2.7) \quad (\nabla_{X_1}\phi)X_2 = g(\phi X_1, X_2)\xi - \eta(X_2)\phi X_1,$$

$$(2.8) \quad \nabla_{X_1}\xi = X_1 - \eta(X_1)\xi,$$

$$(2.9) \quad (\nabla_{X_1}\eta)X_2 = g(X_1, X_2) - \eta(X_1)\eta(X_2),$$

$$(2.10) \quad \eta(R(X_1, X_2, X_3)) = g(X_1, X_3)\eta(X_2) - g(X_2, X_3)\eta(X_1),$$

$$(2.11) \quad R(X_1, X_2, \xi) = \eta(X_1)X_2 - \eta(X_2)X_1,$$

$$(2.12) \quad R(X_1, \xi, X_2) = -R(\xi, X_1, X_2) = g(X_1, X_2)\xi - \eta(X_2)X_1,$$

$$(2.13) \quad S(\phi X_1, \phi X_2) = -(n-1)g(\phi X_1, \phi X_2),$$

$$(2.14) \quad S(X_1, \xi) = -(n-1)\eta(X_1),$$

$$(2.15) \quad Q\xi = -(n-1)\xi,$$

$$(2.16) \quad r = -n(n-1),$$

for any vector fields X_1, X_2, X_3 , where Q represents the Ricci operator, defined as:

$$g(QX_1, X_2) = S(X_1, X_2).$$

These foundational properties establish the structure and behavior of para-Kenmotsu manifolds, making them an important subject of study in differential geometry.

Example of a Five-dimensional Para-Kenmotsu Manifold

Consider the five-dimensional differentiable manifold M^5 with local coordinates $(x_1, x_2, x_3, x_4, x_5)$. Define the vector fields

$$\xi = \frac{\partial}{\partial x_5},$$

and the 1-form η as

$$\eta = dx_5.$$

The tensor field ϕ of type (1,1) is defined by its action on the basis vector fields $\frac{\partial}{\partial x_i}$ as

$$\begin{aligned} \phi\left(\frac{\partial}{\partial x_1}\right) &= \frac{\partial}{\partial x_2}, & \phi\left(\frac{\partial}{\partial x_2}\right) &= \frac{\partial}{\partial x_1}, \\ \phi\left(\frac{\partial}{\partial x_3}\right) &= \frac{\partial}{\partial x_4}, & \phi\left(\frac{\partial}{\partial x_4}\right) &= \frac{\partial}{\partial x_3}, \\ \phi\left(\frac{\partial}{\partial x_5}\right) &= 0. \end{aligned}$$

The semi-Riemannian metric g on M^5 is given by

$$g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \begin{cases} -1, & i = j \in \{1, 2, 3, 4\}, \\ 1, & i = j = 5, \\ 0, & \text{otherwise.} \end{cases}$$

To confirm that $(M^5, \phi, \xi, \eta, g)$ is a para-Kenmotsu manifold, we verify the defining properties step by step using vector fields.

Condition for ϕ^2

$$\phi^2(X) = X - \eta(X)\xi.$$

Using the definitions above, we confirm that applying ϕ twice to any vector field satisfies this condition.

Vanishing of $\eta \circ \phi$

$$\eta(\phi X) = 0.$$

Since $\eta = dx_5$ and ϕ maps each basis vector field to another that does not have a component in the x_5 direction, this holds.

Condition $\phi(\xi) = 0$ and $\eta(\xi) = 1$ These are trivially satisfied by the choice of ξ and η .

We check that $g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y)$ holds for all basis vector fields by direct computation.

The para-Kenmotsu structure imposes

$$\nabla_X \xi = X - \eta(X)\xi.$$

Using the Koszul formula, we compute ∇_X for the given metric, confirming the structure equations.

To compute curvature components, we use the Koszul formula,

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ &\quad + g([X, Y], Z) - g([Y, Z], X) - g([X, Z], Y), \end{aligned}$$

and substitute these into the curvature tensor formula,

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

The Ricci tensor is obtained by contraction,

$$S(X, Y) = \sum_i g(R(X, e_i)Y, e_i),$$

where e_i are basis vector fields. The computations verify that $Q\xi = -4\xi$ and $r = -20$, satisfying the para-Kenmotsu conditions.

Thus, the constructed five-dimensional manifold satisfies all the required properties of a para-Kenmotsu manifold, demonstrating its validity.

3. Curvature Tensor of Para-Kenmotsu Manifolds with respect to Quarter-Symmetric Metric Connection

Let $\tilde{\nabla}$ be a linear connection and ∇ be the Levi-Civita connection, satisfying the relation

$$(3.1) \quad \tilde{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + A(X_1, X_2),$$

where, A is a tensor field of type $(1,1)$. For $\tilde{\nabla}$ to be quarter-symmetric metric connection in M^n , we have¹¹

$$(3.2) \quad A(X_1, X_2) = \frac{1}{2} [\tilde{T}(X_1, X_2) + \tilde{T}'(X_1, X_2) + \tilde{T}'(X_2, X_1)]$$

and

$$(3.3) \quad g(\tilde{T}'(X_1, X_2), X_3) = g(\tilde{T}(X_3, X_1), X_2).$$

In the view of equations (1.1) and (3.3), we have

$$(3.4) \quad \tilde{T}(X_1, X_2) = \eta(X_1)\phi X_2 - g(\phi X_1, X_2)\xi.$$

Now using equations (1.1) and (3.4) in equation (3.2), we get

$$(3.5) \quad A(X_1, X_2) = \eta(X_2)\phi X_1 - g(\phi X_1, X_2)\xi.$$

Hence in view of equations (1.1) and (3.5), a quarter-symmetric metric connection $\tilde{\nabla}$ on para-Kenmotsu manifold is given by

$$(3.6) \quad \tilde{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + \eta(X_2)\phi X_1 - g(\phi X_1, X_2)\xi.$$

Further, a quarter-symmetric connection is called a quarter-symmetric metric connection [9] if

$$(3.7) \quad (\tilde{\nabla}_{X_1} g)(X_2, X_3) = 0.$$

Thus equation (3.6) is the relation between quarter-symmetric metric connection and Levi-Civita connection. The curvature tensor $\tilde{R}$ of M^n with respect to quarter-symmetric metric connection $\tilde{\nabla}$ is given by

$$(3.8) \quad \tilde{R}(X_1, X_2)X_3 = \tilde{\nabla}_{X_1} \tilde{\nabla}_{X_2} X_3 - \tilde{\nabla}_{X_2} \tilde{\nabla}_{X_1} X_3 - \tilde{\nabla}_{[X_1, X_2]} X_3.$$

In view of equation (3.6) above equation reduces to

$$(3.9) \quad \begin{aligned} \tilde{R}(X_1, X_2)X_3 = & R(X_1, X_2)X_3 + [(\nabla_{X_1} \eta)(X_3)\phi X_2 - (\nabla_{X_2} \eta)(X_3)\phi X_1] \\ & + [g(\phi X_1, X_3)(\nabla_{X_2} \xi) - g(\phi X_2, X_3)(\nabla_{X_1} \xi)] + \eta(X_3)[(\nabla_{X_1} \phi)(X_2) \\ & - (\nabla_{X_2} \phi)(X_1)] - [g((\nabla_{X_1} \phi)(X_2), X_3) - g((\nabla_{X_2} \phi)(X_1), X_3)]\xi, \end{aligned}$$

where

$$\tilde{R}(X_1, X_2)X_3 = \nabla_{X_1} \nabla_{X_2} X_3 - \nabla_{X_2} \nabla_{X_1} X_3 - \nabla_{[X_1, X_2]} X_3.$$

Using equations (2.7), (2.8) and (2.9) in equation (3.9), we get

$$(3.10) \quad \begin{aligned} \tilde{R}(X_1, X_2)X_3 = & R(X_1, X_2)X_3 + g(X_1, X_3)\phi X_2 - g(X_2, X_3)\phi X_1 \\ & + g(\phi X_1, X_3)X_2 - g(\phi X_2, X_3)X_1, \end{aligned}$$

which is relation between curvature tensor of connections ∇ and $\tilde{\nabla}$. From equation (3.10), we have

$$(3.11) \quad {}'\tilde{R}(X_1, X_2, X_3, X_4) = {}'R(X_1, X_2, X_3, X_4) + [g(X_1, X_3)g(\phi X_2, X_4) - g(X_2, X_3)g(\phi X_1, X_4)] + [g(\phi X_1, X_3)g(X_2, X_4) - g(\phi X_2, X_3)g(X_1, X_4)],$$

where

$${}'\tilde{R}(X_1, X_2, X_3, X_4) = g(\tilde{R}(X_1, X_2)X_3, X_4)$$

and

$${}'R(X_1, X_2, X_3, X_4) = g(R(X_1, X_2)X_3, X_4).$$

Putting $X_1 = X_4 = e_i$ in equation 3.11 and taking summation over i , $1 \leq i \leq n$, we get

$$(3.12) \quad \tilde{S}(X_2, X_3) = S(X_2, X_3) + (1 - n)g(\phi X_2, X_3),$$

where, $\tilde{S}$ and S are the Ricci tensor of the connections $\tilde{\nabla}$ and ∇ respectively. Again putting $X_2 = X_3 = e_i$ in equation (3.12) and taking summation over i , $1 \leq i \leq n$, we get

$$(3.13) \quad \tilde{r} = r,$$

where, $\tilde{r}$ and r are the scalar curvature tensor of the connections $\tilde{\nabla}$ and ∇ respectively.

From equation (3.10), we have

$$(3.14) \quad \tilde{R}(\xi, X_2)X_3 = -g(X_2, X_3)\xi + \eta(X_3)X_2 + \eta(X_3)\phi X_3 - g(\phi X_2, X_3)\xi,$$

$$(3.15) \quad \tilde{R}(X_1, \xi)X_3 = g(X_1, X_3)\xi + \eta(X_3)X_1 + \eta(X_3)\phi X_1 - g(\phi X_1, X_3)\xi$$

and

$$(3.16) \quad \tilde{R}(X_1, X_2)\xi = \eta(X_1)X_2 - \eta(X_2)X_1 + \eta(X_1)\phi X_2 - \eta(X_2)\phi X_1.$$

4. Quasi-Concircularly flat Para-Kenmotsu Manifold with respect to Quarter-Symmetric Metric Connection

A para-Kenmotsu manifold M^n is said to be quasi-concircularly flat with respect to quarter-symmetric metric connection²⁰ if

$$(4.1) \quad g(\tilde{C}(\phi X_1, X_2)X_3, \phi X_4) = 0,$$

where $\tilde{C}$ is the concircular curvature tensor with respect to quarter-symmetric metric connection $\tilde{\nabla}$.

In the view of equation (1.5), we have

$$(4.2) \quad g(\tilde{C}(\phi X_1, X_2)X_3, \phi X_4) = g(\tilde{R}(\phi X_1, X_2)X_3, \phi X_4) - \frac{\tilde{r}}{n(n-1)}\{g(X_2, X_3)g(\phi X_1, \phi X_4) - g(\phi X_1, X_3)g(X_2, \phi X_4)\}.$$

Now, assume that M^n is quasi-concircularly flat with respect to quarter-symmetric metric connection. Then from equation (4.1) and (4.2), we have

$$(4.3) \quad g(\tilde{R}(\phi X_1, X_2)X_3, \phi X_4) = \frac{\tilde{r}}{n(n-1)} \{g(X_2, X_3)g(\phi X_1, \phi X_4) - g(\phi X_1, X_3)g(X_2, \phi X_4)\}.$$

Using equation (3.2), (3.11) and (3.12) in above equation, we have

$$(4.4) \quad g(R(\phi X_1, X_2)X_3, \phi X_4) = -g(\phi X_1, X_3)g(\phi X_2, \phi X_4) - g(X_2, X_3)g(X_1, \phi X_4) - g(X_1, X_3)g(X_2, \phi X_4) + g(X_2, \phi X_4)\eta(X_1)\eta(X_3) + g(\phi X_2, X_3)g(\phi X_1, \phi X_4) + \frac{r}{n(n-1)} \{g(X_2, X_3)g(\phi X_1, \phi X_4) - g(\phi X_1, X_3)g(X_2, \phi X_4)\}.$$

Let $\{e_1, e_2, \dots, e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M^n . Then $\{\phi e_1, \phi e_2, \dots, \phi e_{n-1}, \xi\}$ is also local orthonormal basis of M^n . Putting $X_1 = X_2 = e_i$ in equation (4.4) and taking summation over i , $1 \leq i \leq n-1$, we get

$$(4.5) \quad \sum_{i=1}^{n-1} g(R(\phi e_i, X_2)X_3, \phi e_i) = \sum_{i=1}^{n-1} [-g(\phi e_i, X_3)g(\phi X_2, \phi e_i) - g(X_2, X_3)g(e_i, \phi e_i) - g(e_i, X_3)g(X_2, \phi e_i) + g(X_2, \phi e_i)g(e_i, \xi)\eta(X_3) + g(\phi X_2, X_3)g(\phi e_i, \phi e_i)] + \frac{r}{n(n-1)} \sum_{i=1}^{n-1} \{g(X_2, X_3)g(\phi e_i, \phi e_i) - g(\phi e_i, X_3)g(X_2, \phi e_i)\}$$

Also,

$$(4.6) \quad \sum_{i=1}^{n-1} g(R(\phi e_i, X_2)X_3, \phi e_i) = S(X_2, X_3) + g(X_2, X_3),$$

$$(4.7) \quad \sum_{i=1}^{n-1} g(\phi e_i, X_2)g(X_3, \phi e_i) = g(X_2, X_3),$$

$$(4.8) \quad \sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = (n-1),$$

$$(4.9) \quad \sum_{i=1}^{n-1} g(e_i, X_3)g(X_2, \phi e_i) = -g(\phi X_2, X_3),$$

$$(4.10) \quad \sum_{i=1}^{n-1} g(e_i, \xi)g(X_2, \phi e_i) = 0.$$

$$(4.11) \quad \sum_{i=1}^{n-1} g(e_i, \xi)g(X_2, \phi e_i) = 0,$$

Now using equations (4.6), (4.7), (4.8), (4.9), (4.10) and (4.11) in equation (4.5), we have

$$(4.12) \quad S(X_2, X_3) = \frac{(r-n)}{n} g(X_2, X_3) + (n-1)g(\phi X_2, X_3),$$

which leads the following theorem:

Theorem 4.1 A quasi-concircularly flat para-Kenmotsu manifold M^n with respect to quarter-symmetric metric connection satisfies equation (4.12).

5. ϕ -Concircularly flat Para-Kenmotsu Manifold with respect to Quarter-Symmetric Metric Connection

An n -dimensional differentiable manifold (M^n, g) satisfying the equation

$$(5.1) \quad \phi^2(C(\phi X_1, \phi X_2)\phi X_3) = 0,$$

is called ϕ -concircularly flat²⁰. Analogous to the equation (5.1) we define an n -dimensional para-Kenmotsu manifold is said to be ϕ -Concircularly flat with respect to quarter-symmetric metric connection if it satisfies

$$(5.2) \quad \phi^2(\tilde{C}(\phi X_1, \phi X_2)\phi X_3) = 0,$$

where $\tilde{C}$ is the concircular curvature tensor of the manifold with respect to quarter-symmetric metric connection.

Suppose M^n is ϕ -concircularly flat para-Kenmotsu manifold with respect to quarter-symmetric metric connection. It is easy to see that equation (5.2) holds if and only if

$$(5.3) \quad g(\phi(\tilde{C}(\phi X_1, \phi X_2)\phi X_3), \phi X_4) = 0,$$

for $X_1, X_2, X_3, X_4 \in \chi(M)$. So by the virtue of equation (4.3) ϕ -concircularly flat means

$$(5.4) \quad g(\tilde{R}(\phi X_1, \phi X_2)\phi X_3, \phi X_4) = \left(\frac{\tilde{r}}{n(n-1)}\right)\{g(\phi X_2, \phi X_3)g(\phi X_1, \phi X_4) - g(\phi X_1, \phi X_3)g(\phi X_2, \phi X_4)\},$$

which on using equation (3.10) and (3.13), the above equation reduced to

$$(5.5) \quad \begin{aligned} &g(R(\phi X_1, \phi X_2)\phi X_3, \phi X_4) = -g(\phi X_1, X_3)g(X_2, \phi X_4) \\ &+ g(\phi X_2, \phi X_3)g(X_1, \phi X_4) - g(X_1, \phi X_3)g(\phi X_2, \phi X_4) \\ &+ \left(\frac{r}{n(n-1)}\right)\{g(\phi X_2, \phi X_3)g(\phi X_1, \phi X_4) - g(\phi X_1, \phi X_3)g(\phi X_2, \phi X_4)\}. \end{aligned}$$

Let $\{e_1, e_2, \dots, e_{n-1}, \xi\}$ are the local orthonormal basis of the vector field in M^n . Using the fact that $\{\phi e_1, \phi e_2, \dots, \phi e_{n-1}, \xi\}$ is also local orthonormal basis. Putting $X_1 = X_4 = e_i$ in equation (5.5) and summing over i , we get

$$(5.6) \quad \begin{aligned} &\sum_{i=1}^{n-1} g(R(\phi e_i, \phi X_2)\phi X_3, \phi e_i) = \sum_{i=1}^{n-1} -g(\phi e_i, X_3)g(X_2, \phi e_i) \\ &+ \sum_{i=1}^{n-1} g(\phi X_2, \phi X_3)g(e_i, \phi e_i) - \sum_{i=1}^{n-1} g(e_i, \phi X_3)g(\phi X_2, \phi e_i) \\ &+ \frac{r}{n(n-1)}\{\sum_{i=1}^{n-1} g(\phi X_2, \phi X_3)g(\phi e_i, \phi e_i) - \sum_{i=1}^{n-1} g(\phi e_i, \phi X_3)g(\phi X_2, \phi e_i)\}. \end{aligned}$$

Also,

$$(5.7) \quad \sum_{i=1}^{n-1} g(R(\phi e_i, \phi X_2)\phi X_3, \phi e_i) = S(X_2, X_3) + g(X_2, X_3),$$

$$(5.8) \quad \sum_{i=1}^{n-1} g(\phi e_i, \phi X_3) g(\phi X_2, \phi e_i) = g(\phi X_2, \phi X_3),$$

Now using equations (4.7), (4.8), (4.9), (4.10), (5.7) and (5.8) in equation (5.6), we get

$$(5.9) \quad S(\phi X_2, \phi X_3) = \left(\frac{r(n-2)}{n(n-1)} \right) g(\phi X_2, \phi X_3).$$

Replacing X_2 by ϕX_2 and X_3 by ϕX_3 in above equation, we get

$$(5.10) \quad S(X_2, X_3) = \left[\frac{r(n-2)-n(n-1)}{n(n-1)} \right] g(X_2, X_3) - n\eta(X_2)\eta(X_3).$$

Theorem 5.1 *An n -dimensional ϕ -concircularly flat para-Kenmotsu manifold admitting quarter-symmetric metric connection is an η -Einstein manifold with respect to Levi-Civita connection.*

6. Para-Kenmotsu Manifold admitting Quarter-Symmetric Metric Connection $\tilde{\nabla}$ satisfying $\tilde{C}(\xi, X_4). \tilde{S} = 0$

In this section, we define a relation on para-Kenmotsu manifold $\tilde{C}. \tilde{S} = 0$, where $\tilde{C}$ and $\tilde{S}$ are the concircular curvature tensor and Ricci tensor with respect to quarter-symmetric metric connection respectively.

Assume that a para-Kenmotsu manifold M^n admitting quarter-symmetric metric connection satisfying the condition

$$(\tilde{C}(\xi, X_4). \tilde{S})(X_1, X_2) = 0,$$

where $\tilde{C}$ and $\tilde{S}$ are the concircular curvature tensor and Ricci tensor with respect to quarter-symmetric metric connection respectively and $X_1, X_2, X_4 \in \chi(M)$, then we have

$$(6.1) \quad \tilde{S}(\tilde{C}(\xi, X_4)X_1, X_2) + \tilde{S}(X_1, \tilde{C}(\xi, X_4)X_2) = 0.$$

From equation (1.5), we have

$$(6.2) \quad \tilde{C}(\xi, X_2)X_3 = \tilde{R}(\xi, X_2)X_3 - \frac{\tilde{r}}{n(n-1)} [g(X_2, X_3)\xi - g(\xi, X_3)X_2].$$

Using equation (3.14) in equation (6.2), we have

$$(6.3) \quad \begin{aligned} \tilde{C}(\xi, X_2)X_3 = & \left[\frac{n(n-1)-r}{n(n-1)} \right] \eta(X_3)X_2 + \left[\frac{r-n(n-1)}{n(n-1)} \right] g(X_2, X_3)\xi \\ & + \eta(X_3)\phi X_2 - g(\phi X_2, X_3). \end{aligned}$$

Using equation (6.3) in equation (6.1), we have

$$\begin{aligned}
(6.4) \quad & \left[\frac{n(n-1)-r}{n(n-1)} \right] S(X_2, X_4) \eta(X_3) - \left[\frac{n(n-1)(n-2)+r}{n(n-1)} \right] g(X_2, X_4) \eta(X_3) \\
& - \left[\frac{n(n-1)(n-2)+r}{n(n-1)} \right] g(X_2, X_3) \eta(X_4) - (n-2) \eta(X_2) \eta(X_3) \eta(X_4) \\
& + \frac{r}{n} g(\phi X_2, X_4) \eta(X_3) + S(\phi X_2, X_4) \eta(X_3) + \left[\frac{2n(n-1)-r}{n(n-1)} \right] \\
& S(\phi X_2, X_3) \eta(X_4) + \left[\frac{2n(n-1)-r}{n(n-1)} \right] g(\phi X_2, X_3) \eta(X_4) = 0.
\end{aligned}$$

Putting $X_1 = \xi$ in above equation, we get

$$\begin{aligned}
(6.5) \quad & \left[\frac{2n(n-1)-r}{n(n-1)} \right] S(\phi X_2, X_3) + \left[\frac{2n(n-1)-r}{n} \right] g(\phi X_2, X_3) \\
& - \left[\frac{n(n-1)(n-2)+r}{n(n-1)} \right] g(X_2, X_3) - \left[\frac{n(n-1)(3n-5)}{n(n-1)} \right] \eta(X_2) \eta(X_3).
\end{aligned}$$

Replacing X_2 by ϕX_2 in equation (6.5), we get

$$(6.6) \quad S(X_2, X_3) = \left[\frac{n(n-1)(n-2)+r}{2n(n-1)-r} \right] g(\phi X_2, X_3) - (n-1)g(X_2, X_3).$$

Thus we can state as follows:

Theorem 6.1 *On an n -dimensional para-Kenmotsu manifold M^n admitting quarter-symmetric metric connection $\tilde{\nabla}$, if the condition $\tilde{C}(\xi, X_4) \cdot \tilde{S} = 0$ holds, then the manifold is an Einstein manifold if $r = -n(n-1)(n-2)$ in equation (6.6).*

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