

Coupled Coincidence Best Proximity Point Theorems on a Metric Space with a Graph

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Abstract: In this paper, we have established the results by introducing the new notion of a coupled coincidence best proximity point instead of the coupled best proximity point in the setting of a partially ordered complete metric space with graph. Our paper is inspired by the work of D. Eshi, P. K Das and P. Debnath¹.

Keywords: Coupled best proximity point; coupled coincidence point; coupled common fixed point; metric space with a graph.

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1. Introduction

One of the most exciting result is the extension of Banach Fixed Point Theorem in complete metric space for non-self mapping as such that A to B does not necessarily has a fixed point where A and B are nonempty closed subsets of a complete metric space (X, d) , with $A \cap B = \emptyset$.

A point x in A for which $d(x, T(x)) = d(A, B)$ is called a best proximity point of T , whenever a non-self mapping T has no optimal approximate solution, where $d(A, B) = \inf \{d(x, y) : x \in A, y \in B\}$. A best proximity point represents an optimal solution to the equation $T(x) = x$. Since best proximity point reduces to a fixed point, if the underlying mapping is assumed to be self - mappings, the Best Proximity Point Theorems are natural deduced of the Banach Fixed Point Theorem.

In this paper, we have deduce the concept of G - g -contraction mapping and proved some coupled coincidence best proximity point theorems for nonlinear contraction mappings in the new set up of complete metric spaces endowed with a directed graph. As an application, we have applied our results to present an existence theorem for solution of some particular integral equations.

In the current paper, our aim is to extend some coupled coincidence best proximity theorems for nonlinear contractions in partially ordered complete metric spaces endowed with a directed graph. Our results would bring about a more unified approach to the presentation of coupled coincidence best proximity and coupled best proximity point theorems.

Let (X, d) be a metric space and Δ be the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ coincides with X and $\Delta \subset E(G)$, where $E(G)$ is the set of edges of the graph G . Assume that G has no parallel edges and so G can be identified with the pair $(V(G), E(G))$. Also, denote by G^{-1} the graph obtained from G by reversing the direction of the edges in G . Thus

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

2. Preliminaries

In this section, we give some basic definitions and concepts related to the main results of this article. In the sequel, N denotes the set of all positive integers and R the set of all real numbers.

For non-empty subsets A and B of a metric space (X, d) , we let

$$d(A, B) = \inf \{d(x, y) : x \in A \text{ and } y \in B\}$$

stands for the distance between A and B .

Now, we introduce the notion of coupled coincidence best proximity point.

Definition 1: Let A and B be nonempty subsets of a metric space (X, d) . An element $(x, y) \in A \times A$ is called a coupled coincidence best proximity point of the function $F: A \times A \rightarrow B$ and $g: X \rightarrow X$ if

$$d(gx, F(x, y)) = d(gy, F(y, x)) = d(A, B).$$

Definition 2:² Let A and B be nonempty subsets of a metric space (X, d) . An element $(x, y) \in A \times A$ is called a coupled best proximity point of the function $F: A \times A \rightarrow B$ if

$$d(x, F(x, y)) = d(y, F(y, x)) = d(A, B).$$

It is easy to see that if $A = B$ in Definition 2, then a coupled best proximity point reduces to a coupled fixed point.

Definition 3:³ The mappings $F: X \times X \rightarrow X$ and $g: X \rightarrow X$ are called commutative if

$$g(F(x, y)) = F(gx, gy),$$

for all $x, y \in X$.

Definition 4:⁴ A function $g: X \rightarrow X$ is G -continuous if

- (i) for all $x, x^* \in X$ and any sequence $\{n_i\}_{i \in \mathbb{N}}$ of positive integers, $\{x_{n_i}\} \rightarrow x^*$ and $(x_{n_i}, x_{n_{i+1}}) \in E(G)$, for all $n \in \mathbb{N}$ implies $g(x) \rightarrow gx^*$;
- (ii) for all $y, y^* \in X$ and any sequence $\{n_i\}_{i \in \mathbb{N}}$ of positive integers, $\{y_{n_i}\} \rightarrow y^*$ and $(y_{n_i}, y_{n_{i+1}}) \in E(G^{-1})$, for all $n \in \mathbb{N}$ implies $g(y) \rightarrow gy^*$.

Definition 5.5 A function $F : X \times X \rightarrow X$ is G -continuous if for all $(x, y), (x^*, y^*) \in X \times X$, and any sequence $\{n_i\}_{i \in \mathbb{N}}$ of positive integers, $\{x_{n_i}\} \rightarrow x^*, \{y_{n_i}\} \rightarrow y^*$ as $i \rightarrow \infty$ and $(x_{n_i}, x_{n_{i+1}}) \in E(G), (y_{n_i}, y_{n_{i+1}}) \in E(G^{-1})$ for $n \in \mathbb{N}$, implies $F(x_{n_i}, y_{n_i}) \rightarrow F(x^*, y^*), F(y_{n_i}, x_{n_i}) \rightarrow F(y^*, x^*)$.

Definition 6.5 Let (X, d) be a complete metric space endowed with a directed graph G . Then the triple (X, d, G) has the property (A) if

- (i) For any sequence $\{x_n\}_{n \in \mathbb{N}}$ in X such that $\{x_n\} \rightarrow x^*$ and $(x_n, x_{n+1}) \in E(G), (x_n, x^*) \in E(G^{-1})$;
- (ii) For any sequence $\{y_n\}_{n \in \mathbb{N}}$ in X such that $\{y_n\} \rightarrow y^*$ and $(y_n, y_{n+1}) \in E(G), (y_n, y^*) \in E(G^{-1})$.

3. Main Results

Now we are ready to discuss our main results. With the help of preliminaries and some definitions given below. First we establish some lemmas which will be useful later. Suppose that A and B be non- empty subset of metric space (X, d) endowed with a directed graph G . Let us consider the mappings $F : A \times A \rightarrow B, F : B \times B \rightarrow A$ and $g : A \rightarrow B$. Define the set $(A \times A)_{Fg}$ and $(B \times B)_{Fg}$ as

$$(A \times A)_{Fg} = \{(x, y) \in X \times X : (gx, F(x, y)) \in E(G) \text{ and } (gy, F(y, x)) \in E(G^{-1})\}.$$

and

$$(B \times B)_{Fg} = \{(y, x) \in X \times X : (gy, F(y, x)) \in E(G) \text{ and } (gx, F(x, y)) \in E(G^{-1})\}.$$

Definition 7. The mapping $F : A \times A \rightarrow B, F : B \times B \rightarrow A$ is called a G - g -contraction if

- (i) G is edge preserving, i.e., $(gx, gu) \in E(G)$ and $(gy, gv) \in E(G^{-1}) \Rightarrow (g(gx), g(gu)) \in E(G)$ and $g(g(y)) \in E(G^{-1})$;
- (ii) F is g -edge preserving, i.e., $(gx, gu) \in E(G)$ and $(gy, gv) \in E(G^{-1}) \Rightarrow (F(x, y), F(u, v)) \in E(G)$ and $F(y, x), F(v, u) \in E(G^{-1})$;
- (iii) for all $x, y \in X$ such that $(gx, gu) \in E(G)$ and $(gy, gv) \in E(G^{-1})$.

The map $F : A \times A \rightarrow B, F : B \times B \rightarrow A$ is said to satisfy g -cyclic contraction for some $\alpha \in [0, 1)$;

$$d(F(x, y), F(u, v)) \leq \frac{\alpha}{2} [d((gx, gu) + d(gy, gv))] + (1 - \alpha)d(A, B)$$

for all $(x, y) \in A \times A, (u, v) \in B \times B$.

Lemma 8. Suppose that A and B be non-empty subsets of a metric space (X, d) . The maps $F: A \times A \rightarrow B, F: B \times B \rightarrow A$ is g -edge preserving and $F(A \times A) \subseteq g(A), F(B \times B) \subseteq g(B)$. Also let $\{x_n\}, \{y_n\}, \{u_n\}, \{v_n\}$ be sequences in the metric space (X, d) endowed with a directed graph G . Then the following statements are true:

(i) $(gx, gu) \in E(G)$ and $(gy, gv) \in E(G^{-1}) \Rightarrow (F(x_n, y_n), F(u_n, v_n)) \in E(G)$ and $(F(y_n, x_n), F(v_n, u_n)) \in E(G^{-1}), \forall n \in \mathbb{N}$;

and

$(gy, gv) \in E(G)$ and $(gx, gu) \in E(G^{-1}) \Rightarrow (F(y_n, x_n), F(v_n, u_n)) \in E(G)$ and $(F(x_n, y_n), F(u_n, v_n)) \in E(G^{-1}), \forall n \in \mathbb{N}$;

(ii) $(x, y) \in (A \times A)_{Fg} \Rightarrow (F(x_n, y_n), F(x_{n+1}, y_{n+1})) \in E(G)$ and $(F(y_n, x_n), F(y_{n+1}, x_{n+1})) \in E(G^{-1}), \forall n \in \mathbb{N}$;

and

$(y, x) \in (B \times B)_{Fg} \Rightarrow (F(y_n, x_n), F(y_{n+1}, x_{n+1})) \in E(G)$ and $(F(x_n, y_n), F(x_{n+1}, y_{n+1})) \in E(G^{-1}), \forall n \in \mathbb{N}$;

(iii) $(x, y) \in (A \times A)_{Fg} \Rightarrow (F(x_n, y_n), F(y_n, x_n)) \in (A \times A)_{Fg}, \forall n \in \mathbb{N}$

and

$(y, x) \in (B \times B)_{Fg} \Rightarrow (F(y_n, x_n), F(x_n, y_n)) \in (B \times B)_{Fg}, \forall n \in \mathbb{N}$.

Proof. We have $F(A \times A) \subseteq g(A)$, set

$$(3.1) \quad F(x_i, y_i) = g(x_{i+1}) \text{ and } F(y_i, x_i) = g(y_{i+1}),$$

$$(3.2) \quad x = x_0, y = y_0$$

(i) Let $(gx, gy) \in Fg$ and $(gy, gx) \in E(G^{-1})$

Then by the g -edge preserving property of F

$$(F(x, y), F(u, v)) \in E(G) \text{ and } (F(y, x), F(v, u)) \in E(G^{-1}) \\ \Rightarrow (gx_1, gu_1) \in E(G) \text{ and } (gy_1, gv_1) \in E(G^{-1}) \text{ (by (3.1))}.$$

Again, applying the g -edge preserving property of F ,

$$(F(x_1, y_1), F(u_1, v_1)) \in E(G) \text{ and } (F(y_1, x_1), F(v_1, u_1)) \in E(G^{-1}).$$

Hence, using (3.1), (3.2) and g -edge preserving property of F repeatedly,

$$(F(x_n, y_n), F(u_n, v_n)) \in E(G) \text{ and } (F(y_n, x_n), F(v_n, u_n)) \in E(G^{-1}) \quad \forall n \in \mathbb{N}.$$

Similarly, we can show that

$$(gy, gv) \in E(G) \text{ and } (gx, gu) \in E(G^{-1}) \Rightarrow (F(y_n, x_n), F(v_n, u_n)) \in E(G) \\ \text{and } (F(x_n, y_n), F(u_n, v_n)) \in E(G^{-1}), \forall n \in \mathbb{N};$$

$$(ii) (x, y) \in (A \times A)_{Fg}$$

$$\Rightarrow (x_0, y_0) \in (A \times A)_{Fg}$$

$$\Rightarrow (gx_0, F(x_0, y_0)) \in E(G) \text{ and } (gy_0, F(y_0, x_0)) \in E(G^{-1}).$$

$$\Rightarrow (gx_0, gx_1) \in E(G) \text{ and } (gy_0, gy_1) \in E(G^{-1}) \text{ (by (3.1)).}$$

Then, by the g -edge preserving property of F

$$(F(x_0, y_0), F(x_1, y_1)) \in E(G) \text{ and } (F(y_0, x_0), F(y_1, x_1)) \in E(G^{-1}).$$

Thus using (3.1) and g -edge preserving property of F repeatedly

$$(F(x_n, y_n), F(x_{n+1}, y_{n+1})) \in E(G) \text{ and } (F(y_n, x_n), F(y_{n+1}, x_{n+1})) \in E(G^{-1}) \forall n \in N.$$

Similarly, we can prove

$$(y, x) \in (B \times B)_{Fg} \Rightarrow (F(y_n, x_n), F(y_{n+1}, x_{n+1})) \in E(G) \text{ and}$$

$$(F(x_n, y_n), F(x_{n+1}, y_{n+1})) \in E(G^{-1}), \forall n \in N;$$

$$(iii) (x, y) \in (A \times A)_{Fg} \Rightarrow F(x_n, y_n), F(x_{n+1}, y_{n+1}) \in E(G) \text{ and}$$

$$F(y_n, x_n), F(y_{n+1}, x_{n+1}) \in E(G^{-1})$$

$$\Rightarrow (gx_{n+1}, F(x_{n+1}, y_{n+1})) \in E(G) \text{ and } (gy_{n+1}, F(y_{n+1}, x_{n+1})) \in E(G^{-1}).$$

(by(3.1))

$$\Rightarrow (x_{n+1}, y_{n+1}) \in (A \times A)_{Fg}.$$

Similarly, we can show that

$$(y, x) \in (B \times B)_{Fg} \Rightarrow (F(y_n, x_n), F(x_n, y_n)) \in (B \times B)_{Fg}, \forall n \in N.$$

Lemma 9: Let A and B be non-empty subsets of a metric space (X, d) . The maps $F: A \times A \rightarrow B$, $F: B \times B \rightarrow A$ be a G - g -contraction with contraction constant $\alpha \in [0, 1)$ and $F(A \times A) \subseteq g(A)$, $F(B \times B) \subseteq g(B)$. Also suppose that $\{x_n\}, \{y_n\}$ be sequences in X . Then, for $(x, y) \in (A \times A)_{Fg}$, $(y, x) \in (B \times B)_{Fg}$, there exist $r(x, y) \geq 0$ such that

$$d(gx_n, gx_{n+1}) \leq \frac{\alpha^{n-1}}{2} r(x, y) \text{ and } d(gy_n, gy_{n+1}) \leq \frac{\alpha^{n-1}}{2} r(x, y).$$

where $r(x, y) = d(gx_0, gx_1) + d(gy_0, gy_1) + (1 - \alpha^{n-1})d(A, B)$.

Proof: $(x, y) \in (A \times A)_{Fg}$

$$\Rightarrow (gx, F(x, y)) \in E(G) \text{ and } (gy, F(y, x)) \in E(G^{-1})$$

$$\Rightarrow (gx_0, gx_1) \in E(G) \text{ and } (gy_0, gy_1) \in E(G^{-1})$$

Then, by Lemma (8), $\forall n \in N$

$$(F(x_n, y_n), F(x_{n+1}, y_{n+1})) \in E(G) \text{ and } (F(y_n, x_n), F(y_{n+1}, x_{n+1})) \in E(G^{-1}),$$

$$\Rightarrow (gx_n, gx_{n+1}) \in E(G) \text{ and } (gy_n, gy_{n+1}) \in E(G^{-1}), \forall n \in N.$$

But F is a G - g -contraction, so

$$\begin{aligned}
d(gx_n, gx_{n+1}) &= d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\
&\leq \frac{\alpha}{2} [d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)] + (1 - \alpha) d(A, B) \\
&= \frac{\alpha}{2} \left[d(F(x_{n-2}, y_{n-2}), F(x_{n-1}, y_{n-1})) + \right. \\
&\quad \left. d(F(y_{n-2}, x_{n-2}), F(y_{n-1}, x_{n-1})) \right] + (1 - \alpha) d(A, B) \\
&\leq \frac{\alpha^2}{2} [d(gx_{n-2}, gx_{n-1}) + d(gy_{n-2}, gy_{n-1})] + \\
&\quad (1 - \alpha^2) d(A, B) \\
&\leq \frac{\alpha^{n-1}}{2} [d(gx_0, gx_1) + d(gy_0, gy_1)] + (1 - \alpha^{n-1}) d(A, B).
\end{aligned}$$

Then $d(gx_n, gx_{n+1}) \leq \frac{\alpha^{n-1}}{2} r(x, y)$, where $r(x, y) = d(gx_0, gx_1) + d(gy_0, gy_1) + (1 - \alpha^{n-1}) d(A, B)$.

Similarly, $d(gy_n, gy_{n+1}) \leq \frac{\alpha^{n-1}}{2} r(x, y)$.

□

Lemma 10: Suppose A and B be nonempty subsets of a complete metric space (X, d) endowed with a directed graph G . Let $F : A \times A \rightarrow B$, $F : B \times B \rightarrow A$ be G - g -contraction with contraction constant $\alpha \in [0, 1)$ and $F(A \times A) \subseteq g(A)$, $F(B \times B) \subseteq g(B)$. Also let $\{x_n\}, \{y_n\}$ be sequences in X . Then, for each $(x, y) \in (A \times A)_{Fg}$, $(y, x) \in (B \times B)_{Fg}$ there exist $x^*, y^* \in X$ such that $gx_n \rightarrow x^*$ and $gy_n \rightarrow y^*$, as $n \rightarrow \infty$.

Proof. Let $(x, y) \in (A \times A)_{Fg}$. Then, using Lemma (9),

$$\begin{aligned}
d(gx_n, gx_{n+1}) &\leq \frac{\alpha^{n-1}}{2} r(x, y) \text{ where } \alpha \in [0, 1), \\
d(gy_n, gy_{n+1}) &\leq \frac{\alpha^{n-1}}{2} r(x, y) \text{ for all } n \in \mathbb{N}.
\end{aligned}$$

Let $m > n$. We have

$$\begin{aligned}
d(gx_n, gx_m) &\leq d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + \cdots + d(gx_{m-1}, gx_m) \\
&\leq \frac{\alpha^{n-1}}{2} r(x, y) + \frac{\alpha^n}{2} r(x, y) + \cdots + \frac{\alpha^{m-2}}{2} r(x, y) \\
&= \frac{\alpha^{n-1}}{2} r(x, y) (1 + \alpha + \cdots + \alpha^{m-n-1}) \\
&= \frac{\alpha^{n-1}}{2} r(x, y) \left(\frac{1 - \alpha^{m-n}}{1 - \alpha} \right) \\
&\Rightarrow d(gx_n, gx_m) \rightarrow d(A, B)
\end{aligned}$$

For large m, n . Hence, $\{gx_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence.

Similarly, $\{g y_n\}_{n \in \mathbb{N}}$ is also a Cauchy sequence. Also, (X, d) is complete. So, there exist, say, $x^*, y^* \in X$ Such that $\lim_{n \rightarrow \infty} g x_n = x^*$ and $\lim_{n \rightarrow \infty} g y_n = y^*$. $\square$

Now we discuss our main results.

Theorem 11: Suppose that A and B be nonempty closed subsets of a complete metric space (X, d) endowed with a directed graph G . Let $F : A \times A \rightarrow B$, $F : B \times B \rightarrow A$ be a G - g -contraction with contraction constant $\alpha \in [0, 1)$ and $F(A \times A) \subseteq g(A)$, $F(B \times B) \subseteq g(B)$. Let g be G -continuous and commutes with F . Also, we assume either

- (a) F is G -continuous, or
- (b) the triple (X, d, G) has the property (A).

Then coupled coincidence best proximity point of Fg is not empty i.e., $CCoinBP(Fg) \neq \emptyset$ iff $(A \times A)_{Fg} \neq \emptyset$ and $(B \times B)_{Fg} \neq \emptyset$.

Proof: Suppose that $CCoinBP(Fg) \neq \emptyset$.

Then there exists some $(x^*, y^*) \in CCoinBP(Fg)$, i.e., $d(gx^*, F(x^*, y^*)) = d(A, B)$ and $d(gy^*, F(y^*, x^*)) = d(A, B)$. So,

$$(gx^*, F(x^*, y^*)) = (gx^* gx^*) \in \Delta \subset E(G)$$

and

$$\begin{aligned} (gy^*, F(y^*, x^*)) &= (gy^*, gy^*) \in \Delta \subset E(G^{-1}) \in \Delta \subset E(G^{-1}) \\ \Rightarrow (x^*, y^*) &\in (A \times A)_{Fg} \\ \Rightarrow (A \times A)_{Fg} &\neq \emptyset, (B \times B)_{Fg} \neq \emptyset \end{aligned}$$

Next, let us assume that $(A \times A)_{Fg} \neq \emptyset, (B \times B)_{Fg} \neq \emptyset$

Then there exists

$$\begin{aligned} (x_0, y_0) &\in (A \times A)_{Fg}, (y_0, x_0) \in (B \times B)_{Fg} \text{ i.e.,} \\ (gx_0, F(x_0, y_0)) &\in E(G) \text{ and } (gy_0, F(y_0, x_0)) \in E(G^{-1}). \end{aligned}$$

Then, by Lemma (8) (ii), we have a sequence $\{n_i\}_{i \in \mathbb{N}}$ of positive integers such that

$$(F(x_{n_i}, y_{n_i}), F(x_{n_{i+1}}, y_{n_{i+1}})) \in E(G) \text{ and } (F(y_{n_i}, x_{n_i}), F(y_{n_{i+1}}, x_{n_{i+1}})) \in E(G^{-1}).$$

Using (3.1) and (3.2),

$$(3.3) \quad (gx_{n_{i+1}}, gx_{n_{i+2}}) \in E(G) \text{ and } (gy_{n_{i+1}}, gy_{n_{i+2}}) \in E(G^{-1})$$

Also, from Lemma (10),

$$(3.4) \quad \lim_{n \rightarrow \infty} gx_{n_i} = x^* \quad \text{and} \quad \lim_{n \rightarrow \infty} gy_{n_i} = y^*$$

But g is G -continuous

$$(3.5) \quad \lim_{n \rightarrow \infty} g(gx_{n_i}) = g(x^*) \quad \text{and} \quad \lim_{n \rightarrow \infty} g(gy_{n_i}) = g(y^*)$$

Also since F and g are commutative,

$$(3.6) \quad \begin{aligned} g(gx_{n_{i+1}}) &= g(F(x_{n_i}, y_{n_i})) \text{ and } g(gy_{n_{i+1}}) = g(F(y_{n_i}, x_{n_i})) \\ (gx_{n_{i+1}}) &= (F(gx_{n_i}, gy_{n_i})) \text{ and } (gy_{n_{i+1}}) = (F(gy_{n_i}, gx_{n_i})) \end{aligned}$$

Finally, we show that $d(gx^*, F(x^*, y^*)) = d(A, B)$ and $d(gy^*, F(y^*, x^*)) = d(A, B)$.

Let F be G -continuous. Then, we have

$$\begin{aligned} d(g(gx_{n_{i+1}}), F(x^*, y^*)) &= d(F(gx_{n_i}, gy_{n_i}), F(x^*, y^*)) \\ &\leq \frac{\alpha}{2} [d(gx_{n_i}, x^*) + (gy_{n_i}, y^*)] + (1 - \alpha)d(A, B) \end{aligned}$$

Taking $i \rightarrow \infty$ we get $d(gx^*, F(x^*, y^*)) = d(A, B)$.

and

$$\begin{aligned} d(g(gy_{n_{i+1}}), F(y^*, x^*)) &= d(F(gy_{n_i}, gx_{n_i}), F(y^*, x^*)) \\ &\leq \frac{\alpha}{2} [d(gy_{n_i}, y^*) + (gx_{n_i}, x^*)] + (1 - \alpha)d(A, B) \end{aligned}$$

Taking $i \rightarrow \infty$ we get $d(gy^*, F(y^*, x^*)) = d(A, B)$.

Thus, (x^*, y^*) is coupled coincidence best proximity point of the mapping F and g , i.e., $CCoinBP(Fg) \neq \emptyset$.

Next, we assume that property (A) holds.

From (3.3) and (3.4), we have $gx_{n_i} \rightarrow x^*$, as $i \rightarrow \infty$ and $(gx_{n_i}, gy_{n_i}) \in E(G)$ and $gy_{n_i} \rightarrow y^*$, as $i \rightarrow \infty$ and $(gy_{n_i}, gx_{n_i}) \in E(G^{-1})$

Therefore using property (A)

$(gx_{n_i}, x^*) \in E(G)$ and $(gy_{n_i}, y^*) \in E(G^{-1})$.

So,

$$\begin{aligned} d(gx^*, F(x^*, y^*)) &\leq d(gx^*, g(gx_{n_{i+1}})) + d(g(gx_{n_{i+1}}), F(x^*, y^*)) \\ &= d(gx^*, g(gx_{n_{i+1}})) + d(F(gx_{n_i}, gy_{n_i}), F(x^*, y^*)) \\ &\leq d(gx^*, g(gx_{n_{i+1}})) + \frac{\alpha}{2} [d(g(gx_{n_i}, gx^*)) + \\ &\quad d(g(gy_{n_i}, gy^*))] + (1 - \alpha)d(A, B). \end{aligned}$$

Now taking the limit as $n \rightarrow \infty$,

$$d(gx^*, F(x^*, y^*)) = d(A, B).$$

In a similar way we can obtain $d(gy^*, F(y^*, x^*)) = d(A, B)$. Hence complete the proof. □

Theorem 12: Suppose that the hypotheses of Theorem (11) hold. Besides, let for every $(x, y), (x^*, y^*) \in A \times A$ and there $(y, x), (y^*, x^*) \in B \times B$ there exist $(u, v) \in X \times X$ such that

$$(F(x, y), F(u, v)) \in E(G), (F(y, x), F(v, u)) \in E(G^{-1}) \text{ and} \\ (F(x^*, y^*), F(u, v)) \in E(G), (F(y^*, x^*), F(v, u)) \in E(G^{-1}).$$

Then F and g have coupled coincidence best proximity point.

Proof. Let (x, y) and (x^*, y^*) be coupled best proximity points, i.e.,

$$(3.7) \quad d(gx, F(x, y)) = d(gy, F(y, x)) = d(A, B)$$

and

$$(3.8) \quad d(gx^*, F(x^*, y^*)) = d(gy^*, F(y^*, x^*)) = d(A, B)$$

By hypothesis, we have

$$(3.9) \quad (F(x, y), F(u, v)) \in E(G), \text{ and } (F(y, x), F(v, u)) \in E(G^{-1})$$

$$(3.10) \quad (F(x^*, y^*), F(u, v)) \in E(G), \text{ and } (F(y^*, x^*), F(v, u)) \in E(G^{-1})$$

Set $F(u_n, v_n) = gu_{n+1}$, $u = u_0$, and $F(v_n, u_n) = gv_{n+1}$, $v = v_0$.

Then, using (3.7) and (3.8); (3.9) and (3.10) can be written as

$$(gx, gu_1) \in E(G), (gy, gv_1) \in E(G^{-1}) \text{ and} \\ (gx^*, gu_1) \in E(G), (gy^*, gv_1) \in E(G^{-1}).$$

But F is g -edge preserving, so

$$(F(x, y), F(u_1, v_1)) \in E(G), (F(y, x), F(v_1, u_1)) \in E(G^{-1}) \text{ and} \\ (F(x^*, y^*), F(u_1, v_1)) \in E(G), (F(y^*, x^*), F(v_1, u_1)) \in E(G^{-1}) \\ \Rightarrow (gx, gu_2) \in E(G), (gy, gv_2) \in E(G^{-1}) \text{ and} \\ (gx^*, gu_2) \in E(G), (gy^*, gv_2) \in E(G^{-1})$$

Using the g -edge preserving property of F repeatedly, $\forall n \geq 1$, we obtain

$$(gx, gu_n) \in E(G), (gy, gv_n) \in E(G^{-1}) \text{ and} \\ (gx^*, gu_n) \in E(G), (gy^*, gv_n) \in E(G^{-1})$$

Therefore,

$$\begin{aligned} d(gx, gx^*) &\leq d(gx, gu_{n+1}) + d(gu_{n+1}, gx^*) \\ &= d(F(x, y), F(u_n, v_n)) + d(F(u_n, v_n), F(x^*, y^*)) \\ &\leq \frac{\alpha}{2} [d(gx, gu_n) + d(gy, gv_n) + d(gu_n, gx^*) + d(gv_n, gy^*)] + \\ &\quad (1 - \alpha)d(A, B) \\ &= \frac{\alpha}{2} [d(F(x, y), F(u_{n-1}, v_{n-1})) + d(F(y, x), F(v_{n-1}, u_{n-1})) \\ &\quad + d(F(u_{n-1}, v_{n-1}), F(x^*, y^*)) \\ &\quad + d(F(v_{n-1}, u_{n-1}), F(y^*, x^*))] + (1 - \alpha)d(A, B) \\ &\leq \frac{\alpha^2}{2} [d(gx, gu_{n-2}) + d(gy, gv_{n-2}) + d(gu_{n-2}, gx^*) + \\ &\quad d(gv_{n-2}, gy^*)] + (1 - \alpha)d(A, B) \end{aligned}$$

$$\leq \frac{\alpha^n}{2} [d(gx, gu_0) + d(gy, gv_0) + d(gu_0, gx^*) + d(gv_0, gy^*)] + (1 - \alpha)d(A, B)$$

$$\Rightarrow d(gx, gx^*) = d(A, B) \quad \text{as } n \rightarrow \infty$$

$$\text{Similarly} \quad d(gy, gy^*) = d(A, B)$$

Let $gx = gx^* = m$ and $gy = gy^* = n$.

Then, using commutativity of F and g , (3.7) gives

$$d(g(gx), g(F(x, y))) = d(gm, F(gx, gy)) = d(m, F(m, n)) = d(A, B) \quad \text{and}$$

$$d(g(gy), g(F(y, x))) = d(gn, F(gy, gx)) = d(n, F(n, m)) = d(A, B)$$

Thus, (m, n) is a coupled coincidence best proximity point.

So, repeating the earlier argument for (x, y) and (m, n) ,

$$gx = gm \quad \text{and} \quad gy = gn \Rightarrow m = gm \quad \text{and} \quad n = gn.$$

Thus,

$$d(m, F(m, n)) = g(gm, (m, n)) = d(A, B) \quad \text{and}$$

$$d(n, F(n, m)) = g(gn, F(n, m)) = d(A, B).$$

So, (m, n) is coupled coincidence best proximity point of F and g .

This complete the proof. □

Example 13: Let $X = \mathbb{R}$ endowed with the usual metric $d(x, y) = |x - y|$. Consider a graph G consisting of $V(G) = X$. Δ be the diagonal of the Cartesian product $X \times X$ and $\Delta \subset E(G)$; where $E(G)$ is the set of edges of the graph G . Also let $A = [1, 5]$ and $B = [-5, -1]$. Clearly, $d(A, B) = 2$. Define $F : A \times A \rightarrow B$, $F : B \times B \rightarrow A$ and $g : A \cup B \rightarrow A \cup B$ by

$$F(x, y, z) = \frac{-x-y-2}{4}, \quad F(u, v, w) = \frac{-u-v+2}{4} \quad \text{and} \quad gx = -x$$

Fixed = $\frac{1}{2}$, for arbitrary $(x, y) \in A \times A$ and $(u, v) \in B \times B$, we get

$$\begin{aligned} d(F(x, y), F(u, v)) &= \left| \frac{-x-y-2}{4} - \frac{-u-v+2}{4} \right| \\ &\leq \frac{|x-u|}{4} + \frac{|y-v|}{4} + 1 \\ &= \frac{\alpha}{2} [d(gx, gu) + d(gy, gv)] + (1 - \alpha)d(A, B). \end{aligned}$$

This implies that F is a g -cyclic contraction pair with $\frac{1}{2}$. Therefore, all hypothesis of our main result hold. So F and g have a coupled coincidence best proximity point. We note that points $(5, 5) \in A \times A$ and $(-5, -5) \in B \times B$ are coupled coincidence best proximity point of F and g . □

4. Coupled Fixed Point Theorem

If we take $A=B$ in the above results i.e., for self-mapping then all results are become corollary¹ and we get unique coupled common fixed point theorem.

Theorem 14: Suppose that (X, d) is a complete metric space endowed with a directed graph G . Let $F: A \times A \rightarrow A$ be a G - g -contraction with contraction constant $\alpha \in [0, \frac{1}{2})$ and $F(A \times A) \subseteq g(A)$. Let g be G -continuous and commutes with F . Also, we assume either

(a) : F is G -continuous, or

(b) : the triple (X, d, G) has the property (A).

Then coupled coincidence point of Fg is not empty i.e., $CCoin(Fg) \neq \emptyset$.

Proof: If we put $d(A, B) = 0$, in Theorem (11). Then we can easily prove that F and g have coupled coincidence fixed point. □

Theorem 15: Suppose that the hypotheses of Theorem (14) hold. Besides, let for every $(x, y), (x^*, y^*) \in A \times A$ there exist $(u, v) \in A \times A$ such that

$(F(x, y), F(u, v)) \in E(G), (F(y, x), F(v, u)) \in E(G^{-1})$ and

$(F(x^*, y^*), F(u, v)) \in E(G), (F(y^*, x^*), F(v, u)) \in E(G^{-1})$.

Then F and g have a unique coupled common fixed point.

Proof: By the use of Theorem (12) we can easily prove the unique coupled common fixed point $(x^*, y^*) \in A \cap B \times A \cap B$ of F and g . □

Example 16. Let $X = \mathbb{R}$ endowed with the usual metric $d(x, y) = |x - y|$. Consider a graph G consisting of $V(G) = X$. Δ be the diagonal of the Cartesian product $X \times X$ and $\Delta \subset E(G)$; where $E(G)$ is the set of edges of the graph G . Also let $A = [-1, 0]$ and $B = [0, 1]$. Define $F: A \times A \rightarrow B$ and $g: A \rightarrow A$ by

$$gx = -x \quad \text{and} \quad F(x, y) = \frac{-(x+y)}{4} \quad \text{and} \quad F(y, x) = \frac{-(y+x)}{4}.$$

Then $d(A, B) = 0$ and F is a g -cyclic contraction with $\alpha = \frac{1}{2}$. Indeed, for $(x, y) \in A \times A$ we have

$$\begin{aligned} d(F(x, y), F(y, x)) &= \left| \frac{-(x+y)}{4} - \frac{-(y+x)}{4} \right| \\ &\leq \frac{|x-y|}{4} + \frac{|y-x|}{4} \\ &= \frac{1}{2} \left(\frac{|x-y|}{4} + \frac{|y-x|}{4} \right) \end{aligned}$$

$$= \frac{\alpha}{2} [d(gx, gy) + d(gy, gx)] + (1 - \alpha)d(A, B)$$

Therefore, all hypothesis of above Theorem holds. So F and g have a unique coupled common fixed point and this point is $(0, 0) \in A \cap B \times A \cap B$. $\square$

5. Conclusion

In this article, we have proved the existence of a coupled coincidence best proximity point for a contraction mapping in a partially ordered complete metric space endowed with a directed graph. Our results bring about a more unified approach to the presentation of coupled coincidence best proximity point theorems because of the use of graph theoretical concepts instead of a partial order.

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