

On Result of Some Nonlinear Functional Integral Equations in Fixed Point Theorem over Banach Algebra

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Abstract: In this paper, we use Lipchitz mapping of the fixed point theorem in Banach algebra to demonstrate the existence of solutions to a few nonlinear functional integral equations. Our results improve and generalize the result of B.C. Dhage¹. An example is also presented.

Keywords: Lipschitzian function, Banach algebra, Compact operator, Banach space.

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1. Introduction

The first time Dhage used fixed point theorems to Banach algebras was in 1988. In several articles, Dhage uses fixed point theorems in Banach algebras to investigate non-linear integral equations. Vijayaraju² proved the validity of the fixed points asymptotic 1-set contraction mapping in real Banach spaces and the sum of two mappings in reflexive Banach spaces. Dhage³ started the research of nonlinear integral equations in Banach algebras using fixed point theorems. This work contains the second significant hybrid fixed point theorem in Banach algebras. Many researchers⁴⁻¹³ have shown some interesting conclusions about the existence of solutions for various nonlinear functional integral equations in Banach algebra.

The purpose of this paper is to prove some existence findings under Lipschitz conditions for a particular set of nonlinear functional integral equations. It is also necessary to prove an application of nonlinear functional-integral equations.

2. Preliminaries

Definition 2.1:³ Let $\mathbb{S}$ be a closed, convex and bounded subset of a Banach algebra $\mathfrak{X}$ and let $\mathcal{P}, \mathcal{Q}: \mathbb{S} \rightarrow \mathbb{S}$ be two operators such that

- (i) $\mathcal{P}$ is Lipschitzian with a Lipschitz constant ρ ,
- (ii) $(\frac{\mathcal{I}}{\mathcal{P}})(a)^{-1}$ exist on $\mathcal{Q}(\mathbb{S})$, where $\mathcal{I}$ is an identity operator and the operator $\frac{\mathcal{I}}{\mathcal{P}}: \mathfrak{X} \rightarrow \mathfrak{X}$ is defined by $(\frac{\mathcal{I}}{\mathcal{P}})(a) = \frac{a}{\mathcal{P}a}$,
- (iii) $\mathcal{Q}$ is completely continuous, and
- (iv) $\mathcal{P}a, \mathcal{Q}b \in \mathbb{S}, \forall a, b \in \mathbb{S}$.

Then the operator equation

$$(1) \quad \mathcal{P}a, \mathcal{Q}a = a$$

has a solution, whenever $\rho\mathfrak{M} < 1$, where $\mathfrak{M} = \|\mathcal{Q}(\mathbb{S})\| = \sup \{\|\mathcal{Q}a\|: a \in \mathbb{S}\}$.

Definition 2.2:⁴ Let $\mathbb{S}$ be a closed, convex and bounded subset of a Banach algebra $\mathfrak{X}$ and let $\mathcal{P}, \mathcal{Q}: \mathbb{S} \rightarrow \mathfrak{X}$ be two operators such that

- (i) $\mathcal{P}$ is Lipschitzian with a Lipschitz constant ρ ,
- (ii) $\mathcal{Q}$ is completely continuous, and
- (iii) $\mathcal{P}a, \mathcal{Q}a \in \mathbb{S}$, for all $a \in \mathbb{S}$.

Then the operator equation $\mathcal{P}a, \mathcal{Q}a = a$ has a solution, whenever $\rho M < 1$, where $M = \|\mathcal{Q}(\mathbb{S})\|$.

Remark 2.3: A mapping $\mathcal{U}: \mathfrak{X} \rightarrow \mathfrak{X}$ is said to be $\mathfrak{D}$ -Lipschitzian, if there exist a continuous and non-decreasing function $\omega: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$(2) \quad \|\mathcal{U}a - \mathcal{U}b\| \leq \omega(\|a - b\|), \text{ for all, } b \in \mathfrak{X}, \text{ where } \omega(0) = 0$$

Sometimes we call the function ω a $\mathfrak{D}$ -function of $\mathcal{U}$ on $\mathfrak{X}$. Obviously every Lipschitzian mapping is $\mathfrak{D}$ -Lipschitzian, but the converse may not be true. If ω is not necessarily non-decreasing and satisfies $\omega(r) < r$, for $r > 0$, the mapping $\mathcal{U}$ is said to be a nonlinear contraction⁵ with a contraction function ω .

Definition 2.4:⁶ Let $\mathbb{S}$ be a closed, convex and bounded subset of a Banach algebra $\mathfrak{X}$ and let $\mathcal{P}: \mathfrak{X} \rightarrow \mathfrak{X}, \mathcal{Q}: \mathbb{S} \rightarrow \mathfrak{X}$ be two operators such that

- (i) $\mathcal{P}$ is $\mathfrak{D}$ -Lipschitzian with a $\mathfrak{D}$ -function ω ,
- (ii) $(\frac{\mathcal{I}}{\mathcal{P}})(a)^{-1}$ exist on $\mathcal{Q}(\mathbb{S})$, where $\mathcal{I}$ is an identity operator and the operator $\frac{\mathcal{I}}{\mathcal{P}}: \mathfrak{X} \rightarrow \mathfrak{X}$ is defined by $(\frac{\mathcal{I}}{\mathcal{P}})(a) = \frac{a}{\mathcal{P}a}$,

(iii) Q is completely continuous,

(iv) $a = Pa.Qb \Rightarrow a \in \mathbb{S}, \forall b \in \mathbb{S}$.

Then the operator equation $a = Pa.Qa$ has a solution, whenever $M\omega(r) < 1, r > 0$, where $M = \|Q(\mathbb{S})\|$.

3. Main Results

Theorem 3.1: Let $\mathbb{S}$ be a closed, convex and bounded subset of a Banach algebra $\mathfrak{X}$ and let $P: \mathfrak{X} \rightarrow \mathfrak{X}, Q: \mathbb{S} \rightarrow \mathfrak{X}, R: \mathfrak{X} \rightarrow \mathfrak{X}$, be three operators such that

(i) P is $\mathfrak{D}$ -Lipschitzian with a $\mathfrak{D}$ -function ω ,

(ii) Q is completely continuous,

(iii) R is $\mathfrak{D}$ -Lipschitzian with a $\mathfrak{D}$ -function ω and completely continuous,

(iv) $a = Pa.Qb + R(c) \Rightarrow a \in \mathbb{S}, \forall b \in \mathbb{S}, c \in \mathbb{S}$.

Then the operator equation $a = Pa.Qa + Ra$ has a solution, whenever $M\omega(r) < 1, r > 0$, where $M = \|Q(\mathbb{S})\|$.

Proof: Let $b, c \in \mathbb{S}$ and define a mapping $P_{bc}: \mathfrak{X} \rightarrow \mathfrak{X}$ by $P_{bc}(a) = Pa.Qb + Rc, a \in \mathfrak{X}$. Here P_{bc} is a nonlinear contraction on $\mathfrak{X}$ with a contraction function φ given by $\varphi(r) = M\omega(r), r \in \mathbb{R}^+$.

We have that

$$\begin{aligned} \|P_{bc}a_1 - P_{bc}a_2\| &\leq \|Pa_1 - Pa_2\| \cdot \|Qb\| \\ &\leq M\omega(\|a_1 - a_2\|), \text{ for all } a_1, a_2 \in \mathfrak{X}. \end{aligned}$$

Now an application of a fixed point theorem of Boyd and Wong⁷ yields that there is a unique point $a^* \in \mathfrak{X}$ such that

$$P_{bc}(a^*) = a^*$$

or, equivalently,

$$a^* = Pa^*.Qb + Rc.$$

Since hypothesis (iv) holds, we have that $a^* \in \mathbb{S}$. Define a mapping $\mathcal{H}: \mathbb{S} \rightarrow \mathfrak{X}$ by

$$(3) \quad \mathcal{H}b = g,$$

where $g \in \mathfrak{X}$ is a unique solution of the equation

$$g = Pg.Qb + Rc; b, c \in \mathbb{S}.$$

We show that $\mathcal{H}$ is continuous. Let $\{b_n\}$ be a sequence in $\mathbb{S}$ converging to a point b . Since $\mathbb{S}$ is closed, $b \in \mathbb{S}$.

Now

$$\begin{aligned} \|\mathcal{H}b_n - \mathcal{H}b\| &= \|(PHb_n.Qb_n + Rc) - (PHb.Qb + Rc)\| \\ &= \|PHb_n.Qb_n - PHb.Qb\| \\ &= \|PHb_n.Qb_n - PHb.Qb_n + PHb.Qb_n - PHb.Qb\| \\ &\leq \|PHb_n.Qb_n - PHb.Qb_n\| + \|PHb.Qb_n - PHb.Qb\| \\ &\leq \|PHb_n - PHb\| \cdot \|Qb_n\| + \|PHb\| \cdot \|b_n - b\| \end{aligned}$$

$$\leq M\omega(\|\mathcal{H}\mathcal{b}_n - \mathcal{H}\mathcal{b}\|) + \|\mathcal{P}\mathcal{H}\mathcal{b}\| \cdot \|\mathcal{b}_n - \mathcal{b}\|$$

$$\therefore \|\mathcal{H}\mathcal{b}_n - \mathcal{H}\mathcal{b}\| \leq M\omega(\|\mathcal{H}\mathcal{b}_n - \mathcal{H}\mathcal{b}\|) + \|\mathcal{P}\mathcal{H}\mathcal{b}\| \cdot \|\mathcal{b}_n - \mathcal{b}\|$$

and hence

$$\limsup_n \|\mathcal{H}\mathcal{b}_n - \mathcal{H}\mathcal{b}\| \leq M\omega(\limsup_n \|\mathcal{H}\mathcal{b}_n - \mathcal{H}\mathcal{b}\|) + \|\mathcal{P}\mathcal{H}\mathcal{b}\|(\limsup_n \|\mathcal{b}_n - \mathcal{b}\|).$$

This shows that $\lim_n \|\mathcal{H}\mathcal{b} - \mathcal{H}\mathcal{b}\| = 0$ and consequently $\mathcal{H}$ is continuous on $\mathbb{S}$. Now we show that $\mathcal{H}$ is compact operator on $\mathbb{S}$.

For any $\mathcal{g} \in \mathbb{S}$, we have

$$\begin{aligned} \|\mathcal{P}\mathcal{g}\| &= \|\mathcal{P}\mathcal{g} - \mathcal{P}x + \mathcal{P}x\| \\ &\leq \|\mathcal{P}\mathcal{g} - \mathcal{P}x\| + \|\mathcal{P}x\| \\ &\leq \|\mathcal{P}x\| + \|\mathcal{P}\mathcal{g} - \mathcal{P}x\| \\ &\leq \|\mathcal{P}x\| + \rho\|\mathcal{g} - x\| \\ &\leq k \\ \therefore \|\mathcal{P}\mathcal{g}\| &\leq k \end{aligned}$$

where $k = \|\mathcal{P}x\| + \text{diam}(\mathbb{S})$ for some fixed $x \in \mathbb{S}$.

Let $\varepsilon > 0$ be given. Since $\mathcal{Q}$ is completely continuous, $\mathcal{Q}(\mathbb{S})$ is totally bounded. Then there is a set $\mathfrak{B} = \{\mathcal{b}_1, \mathcal{b}_2, \dots, \mathcal{b}_n\}$ in $\mathbb{S}$ such that

$$\mathcal{Q}(\mathbb{S}) \subset \bigcup_{i=1}^n B_\delta(v_i),$$

where $v_i = \mathcal{Q}(b_i)$, $\delta = \left(\frac{1 - \rho M}{k}\right)\varepsilon$ and $B_\delta(v_i)$ is an open ball in $\mathfrak{X}$ in centered

at v_i of radius δ . Therefore for any $\mathcal{b} \in \mathbb{S}$ we have a $\mathcal{b}_l \in \mathfrak{B}$ such that

$$\|\mathcal{Q}\mathcal{b} - \mathcal{Q}\mathcal{b}_l\| < \left(\frac{1 - \rho M}{k}\right)\varepsilon.$$

Also we have

$$\begin{aligned} \|\mathcal{H}\mathcal{b} - \mathcal{H}\mathcal{b}_l\| &\leq \|(\mathcal{P}\mathcal{g} \cdot \mathcal{Q}\mathcal{b} + \mathcal{R}c) - (\mathcal{P}\mathcal{g}_l \cdot \mathcal{Q}\mathcal{b}_l + \mathcal{R}c)\| \\ &\leq \|(\mathcal{P}\mathcal{g} \cdot \mathcal{Q}\mathcal{b} + \mathcal{R}c) - (\mathcal{P}\mathcal{g}_l \cdot \mathcal{Q}\mathcal{b} + \mathcal{R}c) + (\mathcal{P}\mathcal{g}_l \cdot \mathcal{Q}\mathcal{b} \\ &\quad + \mathcal{R}c) - (\mathcal{P}\mathcal{g}_l \cdot \mathcal{Q}\mathcal{b}_l + \mathcal{R}c)\| \\ &\leq \|\mathcal{P}\mathcal{g} - \mathcal{P}\mathcal{g}_l\| \cdot \|\mathcal{Q}\mathcal{b}\| + \|\mathcal{P}\mathcal{g}_l\| \cdot \|\mathcal{Q}\mathcal{b}_l - \mathcal{Q}\mathcal{b}\| \\ &\leq (\rho M) \cdot \|\mathcal{g} - \mathcal{g}_l\| + \|\mathcal{P}\mathcal{g}_l\| \cdot \|\mathcal{Q}\mathcal{b}_l - \mathcal{Q}\mathcal{b}\| \\ &\leq \left(\frac{c}{1 - \rho M}\right) \|\mathcal{Q}\mathcal{b}_l - \mathcal{Q}\mathcal{b}\| \\ &\leq \varepsilon \end{aligned}$$

$$\text{i.e. } \|\mathcal{H}\mathcal{b} - \mathcal{H}\mathcal{b}_l\| \leq \varepsilon.$$

This is true for every $\mathcal{b} \in \mathbb{S}$ and hence

$$\mathcal{H}(\mathbb{S}) \subset \bigcup_{i=1}^n B_\varepsilon(\mathcal{g}_i), \text{ where } \mathcal{g}_i = \mathcal{H}(\mathcal{b}_i).$$

As a result $\mathcal{H}(\mathbb{S})$ is totally bounded. Since $\mathcal{H}$ is continuous, it is a compact operator on $\mathbb{S}$.

Now an application of Schauder's fixed point yields that $\mathcal{H}$ has a fixed point in $\mathbb{S}$. Then by the definition of $\mathcal{H}$,

$$a = \mathcal{H}a = \mathcal{P}(\mathcal{H}a).Qa + \mathcal{R}a = \mathcal{P}a.Qa + \mathcal{R}a,$$

and so the operator equation $a = \mathcal{P}a.Qa + \mathcal{R}a$ has a solution in $\mathbb{S}$.

Remark 3.2: Let $\mathbb{S}$ be a closed, convex and bounded subset of a Banach algebra $\mathfrak{X}$ and let $\mathcal{P}: \mathfrak{X} \rightarrow \mathfrak{X}$, $Q: \mathbb{S} \rightarrow \mathfrak{X}$, $\mathcal{R}: \mathfrak{X} \rightarrow \mathfrak{X}$, be three operators such that

- (i) $\mathcal{P}$ is Lipschitzian with a Lipschitzian μ ,
- (ii) Q is completely continuous,
- (iii) $\mathcal{R}$ is Lipschitzian with a Lipschitzian μ and completely continuous
- (iv) $a = \mathcal{P}a.Qb + \mathcal{R}(c) \Rightarrow a \in \mathbb{S}$, $\forall b \in \mathbb{S}, c \in \mathbb{S}$.

Then the operator equation $a = \mathcal{P}a.Qa + \mathcal{R}a$ has a solution, whenever $\mu M < 1$, where $M = \|Q(\mathbb{S})\|$.

Remark 3.3: Let $\mathbb{S}$ be a closed, convex and bounded subset of a Banach algebra $\mathfrak{X}$ such that $\mathbb{S} = \{b \in \mathfrak{X} : \|b\| \leq r\}$ for some real number $r > 0$. Let $\mathcal{P}: \mathfrak{X} \rightarrow \mathfrak{X} - \{0\}$, $Q: \mathbb{S} \rightarrow \mathfrak{X}$, $\mathcal{R}: \mathfrak{X} \rightarrow \mathfrak{X}$ be three operators satisfying

- (i) $\mathcal{P}$ is $\mathfrak{D}$ -Lipschitzian with a $\mathfrak{D}$ -function ω ,
- (ii) Q is completely continuous,
- (iii) $\mathcal{R}$ is $\mathfrak{D}$ -Lipschitzian with a $\mathfrak{D}$ -function ω and completely continuous,
- (iv) $a = \mathcal{P}a.Qb + \mathcal{R}(c) \Rightarrow a \in \mathbb{S}$, $\forall b \in \mathbb{S}, c \in \mathbb{S}$.

Then the operator equation $a = \mathcal{P}a.Qa + \mathcal{R}a$ has a solution, whenever $M\omega(r) < 1$, $r > 0$, where $M = \|Q(\mathbb{S})\|$.

Further, if for all $a, b, c \in \mathfrak{X}$,

$$(4) \quad \|a\| \leq \left\| \left(\frac{I}{P} \right) a \right\|, \|b\| \leq \left\| \left(\frac{I}{P} \right) b \right\|, \|c\| \leq \left\| \left(\frac{I}{P} \right) c \right\|$$

then $a, b, c \in \mathbb{S}$.

Example 3.4: Given a closed and bounded interval $\mathcal{J} = [0, 1]$ in $\mathbb{R}$, the set of all real numbers, consider the nonlinear functional integral equation (FIE)

$$(5) \quad a(t) = \left[\frac{1}{1 + |a(\alpha(t))|} \right] \left(p(t) + \int_0^{\phi(t)} h(x, s(\lambda(x))) dx \right) + \left[\frac{1}{1 + |b(\alpha(t))|} \right]$$

for all $t \in \mathcal{J}$, where $\alpha, \lambda: \mathcal{J} \rightarrow \mathcal{J}$, $p: \mathcal{J} \rightarrow \mathbb{R}$ and $h: \mathcal{J} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous.

By a solution of the FIE (5) we mean a continuous function $a: \mathcal{J} \rightarrow \mathbb{R}$ that satisfies FIE (5) on $\mathcal{J}$.

Let $\mathfrak{X} = \mathfrak{C}(\mathcal{J}, \mathbb{R})$ be a Banach algebra of all continuous real-valued function on $\mathcal{J}$ with the norm

$$(6) \quad \|a\| = \sup_{t \in J} |a(t)|, \quad \|\mathcal{b}\| = \sup_{t \in J} |\mathcal{b}(t)|, \quad \|c\| = \sup_{t \in J} |c(t)|.$$

We shall obtain the solution of FIE (5) under some suitable conditions on the functions involved in (5).

Suppose that the function τ and h satisfy the condition

$$(7) \quad \tau(t) \leq t, \quad |h(t, a)| < 1 - \|p\|, \quad \|p\| < 1, \quad \text{for all } t \in J \text{ and } a \in \mathbb{R}.$$

Define a subset $\mathbb{S}$ of $\mathfrak{X}$ by

$$(8) \quad \mathbb{S} = \{\mathcal{b} \in \mathfrak{X} : \|\mathcal{b}\| \leq 1\}.$$

Consider the three mappings $\mathcal{P}, \mathcal{Q}, \mathcal{R}: \mathfrak{X} \rightarrow \mathfrak{X}$ defined by

$$(9) \quad \mathcal{P}a(t) = \frac{1}{1 + |a(\alpha(t))|}, \quad t \in J, \quad \mathcal{Q}a(t) = p(t) + \int_0^{\phi(t)} h(x, s(\lambda(x))) dx, \quad t \in J,$$

$$\text{and } \mathcal{R}a(t) = \frac{1}{1 + |b(\alpha(t))|}, \quad t \in J.$$

Then the FIE (4) is equivalent to the operator equation

$$(10) \quad \mathcal{P}a(t) \cdot \mathcal{Q}a(t) + \mathcal{R}a(t) = a(t), \quad t \in J.$$

We shall show that the operators $\mathcal{P}, \mathcal{Q}$ and $\mathcal{R}$ satisfy all the conditions of remark 2.1. Clearly $\mathcal{P}$ define a mapping $\mathcal{P}: \mathfrak{X} \rightarrow \mathfrak{X} - \{0\}$. First we show that $\mathcal{P}$ is Lipschitzian on $\mathfrak{X}$. Let $a, \mathcal{b}, c \in \mathfrak{X}$. Then we have

$$\begin{aligned} |\mathcal{P}a(t) - \mathcal{P}\mathcal{b}(t)| &= |1/(1 + |a(\alpha(t))|) - 1/(1 + |\mathcal{b}(\alpha(t))|)| \\ &= \frac{||a(t)| - |\mathcal{b}(t)||}{(1 + |a(\alpha(t))|)(1 + |\mathcal{b}(\alpha(t))|)} \\ &\leq \|a - \mathcal{b}\|. \end{aligned}$$

Taking the supremum over t , we obtain

$$\|\mathcal{P}a - \mathcal{P}\mathcal{b}\| \leq \|a - \mathcal{b}\|,$$

which shows that $\mathcal{P}$ is a Lipschitzian with a Lipschitz constant $\rho = 1$.

To prove $\mathcal{Q}$ is completely continuous on $\mathbb{S}$, for that we show $\mathcal{Q}: \mathbb{S} \rightarrow \mathbb{S}$.

Let $a \in \mathbb{S}$, then by (7) and (9)

$$\begin{aligned} |\mathcal{Q}a(t)| &\leq |p(t)| + \int_0^{\phi(t)} |h(x, s(\lambda(x)))| dx \\ &< |p(t)| + \int_0^t (1 - \|q\|) dx \end{aligned}$$

Since $\mathcal{Q}a \in \mathfrak{C}(J, \mathbb{R})$, there is a point $t_1 \in J$ such that

$$\|\mathcal{Q}a\| = |\mathcal{Q}a(t_1)| = \max_{t \in J} |\mathcal{Q}a(t)|.$$

Therefore, we have

$$\|\mathcal{Q}a\| = |\mathcal{Q}a(t_1)| < |p(t_1)| + \int_0^{t_1} (1 - \|q\|) dx \leq \|p\| + \int_0^1 (1 - \|q\|) dx = 1,$$

i.e. $\|Qa\| < 1$. As a result $Q: \mathbb{S} \rightarrow \mathbb{S}$. Similarly we can show that $\mathcal{R}$ is $\mathfrak{D}$ -Lipschitzian with a $\mathfrak{D}$ -function ω and completely continuous. Finally we show that conditions of remark 2.3 holds.

Now for any $a \in \mathfrak{X}$,

$$\left\| \left(\frac{I}{P} \right)(a) \right\| = \sup_{t \in J} \left| \frac{a(t)}{\mathcal{P}a(t)} \right| = \sup_{t \in J} \{ |a(t)| [1 + |a(\alpha(t))|] \} \geq \|a\|$$

Similarly,

$$\left\| \left(\frac{I}{P} \right)(b) \right\| \geq \|b\|, \quad \left\| \left(\frac{I}{P} \right)(c) \right\| \geq \|c\|,$$

and so by remark 2.3, condition (iv) of Theorem 2.1 is satisfied. Thus operator $\mathcal{P}$, Q , $\mathcal{R}$ satisfy all the conditions of Theorem 2.1 and hence an application of it yields that the operator equation (10) and consequently the FIE (5) has a solution on J .

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