

A Generalized (λ, μ) - Slant Hankel Operators on the Lebesgue Space $L^2(T)$

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Abstract: In this paper, we shall look for the solution of an operator equation $\mu M_z^* X - X M_z^k = \lambda X$, for $k \geq 1$ and $\lambda, \mu \in \mathbb{C}$. Its solutions will provide a class of operators, which can be considered as generalization of λ -Hankel operator (introduced by Avendano). We also analyze certain elementary properties of generalized (λ, μ) - slant Hankel operator on the Lebesgue space of unit circle T . We also discuss an analogous result to the famous Kronecker theorem of Hankel matrix for generalized (λ, μ) - slant Hankel operator.

Keywords: Toeplitz operator, Hankel operator, Slant Hankel operator, Lebesgue space, Hardy space, λ -Hankel operator, λ -Slant Hankel operator.

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1. Introduction

The study of Toeplitz operators started in the beginning of the twentieth century due to Toeplitz¹⁻³, with the investigation of the relationship of finite square matrices constant along diagonals, to the corresponding infinite matrices. More formally, a matrix $(a_{m, n})$ is a Toeplitz matrix if and only if $(a_{m_1, n_1}) = (a_{m_2, n_2})$, whenever $m_1 - n_1 = m_2 - n_2$. Toeplitz operator can be considered as compression of multiplication operator to the Hardy space $H^2(D)$ and it can be characterized by $U^* T U = T$, where U is the unilateral shift operator.

The most important counterpart of Toeplitz operator is the Hankel operator. In his doctoral dissertation, Hankel introduced the study of finite Hankel matrices. The Kronecker theorem was the first recent development towards infinite Hankel matrices. A matrix $(a_{m, n})$ is a Hankel matrix if $(a_{m_1, n_1}) = (a_{m_2, n_2})$, whenever $m_1 + n_1 = m_2 + n_2$. Hankel operator³⁻⁵ can be considered as compression of flipped multiplication operator $J M \phi$ to the Hardy

space $H^2(T)$, where the operator $J : L^2(T) \rightarrow L^2(T)$ is the flip operator given by $Jf(z) = f(\bar{z})$. Few fundamental theorems namely, Douglas theorem, the Julia-Halmos theorem, Parrott's theorem address the question: "when do a bounded operator with Hankel matrix represent Hankel operator". Nehari theorem gives a condition for this question and states that a Hankel matrix corresponds to a bounded operator H on the Hardy space if and only if there exists φ in $L^\infty(T)$ such that $H = H_\varphi = PJM_\varphi$. One of the milestone theorems in the study of Hankel operator is the Kronecker's Theorem, which states that the Hankel operator H_φ is of finite rank if and only if $\varphi(z) = \sum_{n=0}^{\infty} a_{-n}z^{-n}$ is a rational function, whose poles lies in D and φ in $L^\infty(T)$. The compactness problem related to Hankel operator was analyzed by Hartman⁶ in his classical theorem, which states that a Hankel operator H is compact if and only if there exists a continuous function φ such that $H = H_\varphi$." We also know that every compact Hankel operator is uniform limit of finite-rank Hankel operators.

The area of Toeplitz operators and Hankel operators have been a field of extensive research. These operators produce two of the most extensive and interesting classes of operators to the Hardy spaces. Brown and Halmos² work towards properties of Toeplitz operators, which acts as a foundation stone in the progress of the study of Toeplitz operators. The theory of Toeplitz operators and matrices has links with many branches of Mathematics (specially Operator Theory), namely, C^* -algebras, Function Theory, Prediction Theory (see papers^{7,8} and the references therein).

The operator whose matrix with respect to an orthonormal basis can be obtained by deleting alternating rows of the corresponding Toeplitz or Multiplication operator is called a slant Toeplitz operator. The class of slant Toeplitz operators is one of the significant class of operators on the Lebesgue space $L^2(T)$. A systematic study of class of slant Toeplitz operators started in 1995 by M.C. Ho⁹⁻¹². It is easy to see that there is a close connection among slant Toeplitz operators, Multiplication or Laurent operators, and composition operators. As a matter of the fact, each slant Toeplitz operator S_φ on the Lebesgue space $L^2(T)$, can be written as WM_φ , where M_φ is the Laurent operator on $L^2(T)$ given by the essentially bounded measurable function φ on the unit circle T , and

$$W(z^n) = \begin{cases} z^{\frac{n}{2}}, & \text{when } n \text{ is even} \\ 0, & \text{when } n \text{ is odd.} \end{cases}$$

The adjoint of slant Toeplitz operator is a weighted composition operator. There are several researchers (namely, Arora, Batra, Datt, Agarwal, Ohri, Pandey and Hazarika etc.) who are actively working and investigating several properties of slant Toeplitz operators on the Lebesgue space of certain measure spaces (mainly on the unit circle). It has several applications in

ergodic theory, dynamical systems and wavelet analysis (see papers^{8,13-15}). S.C. Arora, R. Batra, and M.P. Singh¹⁶ have introduced the notion of slant Hankel operators and investigated several algebraic and spectral properties. They proved that spectrum of slant Hankel operator K_ϕ contains a closed disc for an invertible symbol ϕ .

The study on weighted sequence spaces $L^2(\beta)$ and $H^2(\beta)$, introduced by Kelley¹⁷, is a notable area of research in Operator theory. For a sequence $\beta = (\beta_n)$ of positive real numbers, the space $L^2(\beta)$ consists of all formal Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$ satisfying the condition $\|f\|_\beta^2 = \sum_{n=-\infty}^{\infty} |a_n|^2 \beta_n < \infty$, where $\mathbb{C}$ is the set of all complex numbers. Obviously, it is a Hilbert space. Similarly, the space $H^2(\beta)$ consists of all formal power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$ with $\|f\|_\beta^2 < \infty$. It is a subspace of space $L^2(\beta)$. The space of all formal Laurent series $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, such that $\phi \in L^2(\beta) \subseteq L^2(\beta)$ and $\|\phi f\|_\beta < c \|f\|_\beta$ for each $f \in L^2(\beta)$ and for some $c > 0$, is represented by $L^\omega(\beta)$, whereas the space of all formal power series $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$, such that $\phi \in H^2(\beta) \subseteq H^2(\beta)$ is denoted by $H^\omega(\beta)$. These spaces reduce to certain famous Hilbert spaces consisting of analytic functions corresponding to particular choices for sequence $\beta = (\beta_n)$. For instance, if we choose $\beta_n = 1$ for all $n \geq 0$, then the space $H^2(\beta)$ yields the famous classical Hardy space $H^2(D)$. For $\beta_n = (n+1)^{\frac{1}{2}}$ and $\beta_n = (n+1)^{-\frac{1}{2}}$ the space $H^2(\beta)$ corresponds to Dirichlet space and Bergman space respectively.

Therefore, the study of operators on such spaces is more enlightening. A systematic investigation of weighted shift operators on these spaces was done by Kelley and these operators have been studied thoroughly and now it appears as examples and counter examples in literature.

It is easy to observe that the operator equations act as a handy tool to characterize certain class of operators and to generalize them to the next level. Barria and Halmos² asked for a solution of operator equation $U^*XU = \lambda X$, for a complex number λ , which is solved by Sun⁸ completely and gave a generalization of Toeplitz operators (namely, λ - Toeplitz operator) on $H^2(T)$. They also considered the operator equation $U^*XU - X = K$ for a compact operator K on the Hardy space $H^2(T)$, which characterizes essentially Toeplitz operators. These studies have led to various generalizations of Toeplitz and Hankel operators. Avendano^{4,5} introduced λ -Hankel operators and essentially Hankel operators. The operators (namely, slant Toeplitz operators, λ -Hankel operators and λ - Toeplitz operators) are further uplifted using the Calkin algebras $\frac{B(L^2)}{K(L^2)}$ and $\frac{B(H^2)}{K(H^2)}$ (see papers^{4,5}), where $B(H)$ and $K(H)$ denote the spaces of all bounded linear operators on H , and all compact operators on

Hilbert space H respectively. These studies were further enriched with the notion of another variant of λ -slant Hankel operators¹⁸ (namely, non-spectral problem), essentially (λ, μ) -Hankel operators¹⁹.

The Toeplitz and Hankel operators have also been discussed on different function spaces. Gu and Zheng²⁰ taken Toeplitz operators into consideration on the Hardy space of the bidisk. H. Guediri²¹ has initiated the notion of dual Toeplitz operators on the sphere and has further uplifted the notion of Toeplitz and Hankel operators on the Hardy space of unit sphere^{22,23}. Operators related Hankel and Toeplitz operator on model spaces are recently considered^{24,25}.

2. Generalized (λ, μ) - Slant Hankel Operator

Here, we introduce (λ, μ) - k th-order slant Hankel operator which can be characterize by an operator equation motivated by the work of Barria, Halmos, Avendano and Arora. Initially, we provide a list of few definitions and results.

Definition 2.1 (k th-order slant Hankel operator)^{1,16} A k th-order slant Hankel operator on the Lebesgue space is given by $S\phi = W_k J M \phi$, where $\phi \in L^\infty(T)$, J is the flip operator on $L^2(T)$, $M\phi$ is multiplication operator induced by ϕ and

$$W_k(z^n) = \begin{cases} z^{\frac{n}{k}}, & \text{when } n \text{ is a multiple of } k \\ 0, & \text{otherwise.} \end{cases}$$

We know that these operator on the space $L^2(T)$ has been characterized by the operator equation $M_z^* X = X M_z^k$.

Definition 2.2 (λ - slant Hankel operator)¹⁴ A solution of the non-spectral type operator equation $\lambda M_z^* X = X M_z^k$ is given by $S_{\phi, \lambda} = D_\lambda W_k J M \phi$, which is called as λ - slant Hankel operator on $L^2(T)$. It is induced by a function ϕ of $L^\infty(T)$ and $D_\lambda(f(z)) = f(\bar{\lambda}z)$. The operators W_k , J (flip operator), and $M\phi$ (multiplication operator) are same as defined above.

Next, certain important results are listed, which are useful in our discussion. It is well known that every Hilbert space can be visualized as a functional Hilbert space⁹ (that is, as a Hilbert space H containing complex valued functions on a non-empty set Y such that the evaluation functional $\phi_y: H \rightarrow \mathbb{C}$ is bounded as well as linear for all $y \in Y$). A reproducing kernel of functional Hilbert space H are given by the following theorem in terms of an orthonormal basis of the Hilbert space H .

Theorem 2.3.⁹ Let H be a functional Hilbert space over Y then the reproducing kernel of H is given by

$$K(x; y) = K_y(x) = \sum_n e_n(x) e_n^*(y),$$

where (e_j) be an orthonormal basis of the space H and $x, y \in Y$.

It is trivial from the preceding theorem that a function $S_w: D \rightarrow \mathbb{C}$ given by

$$(2.1) \quad S_w(z) = \sum_n e_n(z) e_n^*(w) = \sum_n z^n \overline{w}^n,$$

defines a reproducing kernel for the Hilbert space $L^2(T)$. Next, we are ready to initiate the analysis of generalized (λ, μ) - slant Hankel operator. For this, let us take following operator equation into consideration

$$(2.2) \quad \mu M_z^* T - T M_z^k = \lambda T$$

for complex numbers μ, λ and $k \geq 2$. In this paper, we are looking for bounded operators T on the Lebesgue space satisfying the operator equation (2.2) for fixed numbers $\mu, \lambda \in \mathbb{C}$. These operators $T \in B(L^2)$ are termed as generalized (λ, μ) - slant Hankel operator.

Depending on the positions of the constants $\mu, \lambda \in \mathbb{C}$, there could be another symmetrical form $\mu M_z^* T - \lambda T M_z^k = T$ of the given equation (2.2). One can easily observe that the operator equation, given in (2.2), have more non-trivial solutions than the operator equation $\mu M_z^* T - \lambda T M_z^k = T$.

The following outputs can be drawn by assigning particular values to the constants $k \geq 1$ and $\mu, \lambda \in \mathbb{C}$ in the equation (2.2).

- For the choices $k = 2$, and $\mu = 1$, the equation (2.2) resulted into a spectral type problem $M_z^* T - T M_z^2 = \lambda T$. It characterizes λ - slant Hankel operator.
- For $k \geq 2$, and $\mu = 1$, the equation (2.2) again provides a spectral type problem $M_z^* T - T M_z^k = \lambda T$, whose solutions are known as generalized λ -slant Hankel operator.
- For $k = 2$, $\lambda = 0$, the operator equation (2.2) reduces to $\mu M_z^* T - T M_z^2 = 0$. It gives another form of μ - slant Hankel operator on $L^2(T)$.
- For $k \geq 2$, $\lambda = 0$, the operator equation (2.2) reduces to $\mu M_z^* T - T M_z^k = 0$, which provides generalized μ -slant Hankel operator on the Lebesgue space $L^2(T)$.
- The choices $k = 2$ and $(\lambda, \mu) = (0; 1)$ in the equation (2.2) yields $M_z^* T - T M_z^2 = 0$, which characterizes slant Hankel operator.

Next, we ask for solution of the operator equation $\mu M_z^k X - X M_z^* = \lambda X$ for fixed complex numbers $\mu, \lambda \in \mathbb{C}$.

Theorem 2.4. *There is only trivial solution for an operator equation $\mu M_z^k X - X M_z^* = \lambda X$ for fixed complex numbers $\mu, \lambda \in \mathbb{C}$ and $k \geq 2$.*

Proof: For $k \geq 2$ and $\mu, \lambda \in \mathbb{C}$, consider the equation $\mu M_z^k X - X M_z^* = \lambda X$, which may be written as $(\mu M_z^k - \lambda I)X = X M_z^*$. Let $f \in \ker(\mu M_z^k - \lambda I)$ and f has the form $f(z) = \sum_{m=-\infty}^{\infty} a_m z^m$, where $\ker(T)$ denotes the kernel of the operator T . Then, it implies that

$$\begin{aligned}
(\mu M_z^k - \lambda I)f(z) &= 0 \\
(\mu M_z^k - \lambda I)(\sum_{m=-\infty}^{\infty} a_m z^m) &= 0 \\
\sum_{m=-\infty}^{\infty} \mu a_m z^{m+k} &= \sum_{m=-\infty}^{\infty} \lambda a_m z^m.
\end{aligned}$$

Now, we replace m by $m - k$ on the left-hand side of the above equation. We have

$$\sum_{m=-\infty}^{\infty} \mu a_{m-k} z^m = \sum_{m=-\infty}^{\infty} \lambda a_m z^m.$$

Comparing the corresponding coefficients in the preceding equation, we obtain that

$$(2.3) \quad \lambda a_m = \mu a_{m-k}; \text{ for } k \geq 2.$$

From the above expression, it follows that,

Case I: If $\lambda = 0$ and $\mu \neq 0$, then the relation (2.3) turns out that $a_{m-k} = 0$ for all $m \in \mathbb{Z}$ and $k \geq 2$. On replacing m by $m + k$, it follows that each coefficient a_m of f is equal to 0, which turns out that $f = 0$.

Case II: When $\lambda \neq 0$ and $\mu = 0$, then the above relation (2.3) trivially yields that $a_m = 0$ for all $m \in \mathbb{Z}$, and hence $f = 0$.

Case III: If both λ and μ are non-zero, then using the relation (2.3) repeatedly, we obtain that $a_{km+i} = c^m a_i$, for $m \in \mathbb{Z}$ and $i \in \{0, 1, 2, \dots, k-1\}$, where $c = \frac{\mu}{\lambda}$. As we know that for any function $f \in L^2(T)$, it can be written as $f(z) = \sum_{m=-\infty}^{\infty} a_m z^m$ with the condition $\sum_{m=-\infty}^{\infty} |a_m|^2 < \infty$. It gives that for each $i \in \{0, 1, 2, \dots, k-1\}$, the series given by

$$\sum_{m=-\infty}^{\infty} |a_{km+i}|^2$$

is convergent. This is true only if the coefficients $a_{km+i} = 0$ for each $m \in \mathbb{Z}$ and $0 \leq i \leq k-1$. It ultimately reduces to $a_m = 0$ for all $m \in \mathbb{Z}$, which gives that $f = 0$ and hence $\ker((\mu M_z^k - \lambda I)) = \{0\}$. Now, by the mean of Theorem 2.14, we conclude that $X = 0$.

Case IV: For the case when $(\lambda, \mu) = (0; 0)$, the equation $\mu M_z^k X - X M_z^* = \lambda X$ readily gives that $X M_z^* = 0$, which ensures that $X = 0$.

Next, we approach to our main aim and revisit the equation

$$(2.4) \quad \mu M_z^* T - T M_z^k = \lambda T$$

in the search of solution. It is well known that the μ -slant Hankel operator for a fixed complex number μ is studied by Datt and Aggarwal¹⁴, which is a particular case of considered operator equation for $\lambda = 0$ and $k = 2$. It can be readily deduced that the operator equation obtained by assigning $\lambda = 0$ and $\mu = 0$ in the considered operator equation admits only zero solution. The following result furnishes the solution of operator equation $\mu M_z^* T - T M_z^k = \lambda T$ for $\mu = 0$.

Theorem 2.5. *Let λ be a fixed complex number. Then, the operator equation $-XM_z^k = \lambda X$ admits only zero solution.*

Proof. The given operator equation $-XM_z^k = \lambda X$ is rephrased as

$$(2.5) \quad X(M_z^k + \lambda I) = 0.$$

It is easy to check that the spectrum of M_z^k is unit circle T . Consequently, for the choice of λ such that $|\lambda| \neq 1$, the operator $(M_z^k + \lambda I)$ resulted into non-singular operator, which enforces that the operator equation (2.5) admits trivial solution.

For the case, when the complex number λ is unimodular, equation (2.5) reduces to $X = cXM_z^k$, where $c = -\lambda$. As a consequence, it yields that $X(z^n) = cX(z^{n-k})$ for each integer n . Employing this relation repeatedly, we produce certain relations in terms of basis elements $\{e_m(z) = z^m\}$ of $L^2(T)$, that is,

$$X(z^{km+i}) = c^m X(z^i) \text{ for } m \in \mathbb{Z} \text{ and } i \in \{0, 1, 2, \dots, k-1\}.$$

Let if possible that there exists at least one $i \in \{0, 1, 2, \dots, k-1\}$ such that $X(z^i)$ is non-zero function. It immediately allows us to choose a non-zero coefficient a_j in the Fourier expansion of $X(z^i)$ for an integer j . Let us choose a sequence (s_n) of elements of j th row of the matrix representation of X . It is noticed that (s_n) does not converge to zero, as it has a subsequence (s_{n_k}) where $n_k = kt$ for some integer t , which does not converge to 0. It shows in contradiction to the fact that the given operator X is bounded. Therefore, it follows that $X(z^i) = 0$, for each $i \in \{0, 1, \dots, k-1\}$ and hence $X(z^n) = 0$ for all n . Hence the result follows.

It is easy to observe that for a given generalized (λ, μ) - slant Hankel operator T and a multiplication operator $M\phi$ on $L^2(T)$, we have

$$\mu M_z^* (T M\phi) - (T M\phi) M_z^k = (\mu M_z^* T - T M_z^k) M\phi = \lambda T M\phi$$

and

$$\mu M_z^* (M\phi T) - (M\phi T) M_z^k = M\phi (\mu M_z^* T - T M_z^k) = \lambda M\phi T$$

It readily yields that the operators $T M\phi$ and $M\phi T$ are generalized (λ, μ) -slant Hankel operators. Therefore, we conclude that the class (λ, μ) -SHank $_k(L^2)$ is closed under the composition of $M\phi$ for each $\phi \in L^\infty(T)$. Now, we going to discuss a spectral property of elements in the class (λ, μ) -SHank $_k(L^2)$, which is extremely important.

Theorem 2.6. *For a given element T from the class (λ, μ) - SHank $_k(L^2)$, both the spectrum $\sigma(T)$ and essential spectrum $\sigma_e(T)$ of T contains 0.*

Proof: Suppose that T is an arbitrary element of the class (λ, μ) -SHank $_k(L^2)$. It is well known that the zero operator is not invertible as well as not essentially invertible. Therefore, T is not invertible as well as not essentially invertible, whenever $T = 0$. That is why let us take T to be a non-zero invertible operator satisfying $(\mu M_z^* - \lambda I) T = T M_z^k$. It follows that the

operators M_z^k and $(\mu M_z^* - \lambda I)$ are similar operators. It is a contradiction. Therefore, T is not invertible operator.

In the similar manner, suppose that X is a non-zero essentially invertible operator then the relation $(\mu M_z^* - \lambda I) X = X M_z^k$ provides two essentially similar operators (namely, M_z^k and $(\mu M_z^* - \lambda I)$). But these operators are not essentially similar operators. Therefore, our assumption was false. This completes the proof.

Now, we provide a similar result as given by Avandano^{4,5}.

Lemma 2.7. *Let A be bounded operator on $L^2(T)$ with $\|A\| < 1$. Then the equation $AX = XM\phi$ admits only zero solution for $\phi \in L^\infty(T)$ with $\|\phi\|_\infty < 1$.*

Consequently, we derive following result with the aid of preceding lemma.

Corollary 2.8. *For $|\mu| + |\lambda| < 1$, the zero operator ($X = 0$) is the only operator satisfying the equations $\mu M_z^* X - X M_z^k = \lambda X$ and $\mu M_z X - X M_z^k = \lambda X$.*

Proof. Let X be solution of equation $\mu M_z^* X - X M_z^k = \lambda X$ and $\mu M_z X - X M_z^k = \lambda X$. It follows that the operator X satisfies the operator equation $(\mu M_z^* - \lambda I) X = X M_z^k$ and $(\mu M_z - \lambda I) X = X M_z^k$. It is easy to see that the norms of operators $(\mu M_z^* - \lambda I)$ and $(\mu M_z - \lambda I)$ satisfies the inequality $\|(\mu M_z^* - \lambda I)\| < |\mu| + |\lambda| < 1$ and $\|(\mu M_z - \lambda I)\| < |\mu| + |\lambda| < 1$, respectively. With the aid of above lemma, it is easily obtained that both the operator equations admit only trivial solution.

The following result provides condition under which a generalized (λ, μ) -slant Hankel operator is a generalized (α, β) -slant Hankel operator for $\alpha, \beta, \lambda, \mu \in \mathbb{C}$.

Theorem 2.9. *The zero operator is the only operator, which is generalized (λ, μ) -slant Hankel operator as well as generalized (α, β) -slant Hankel operator.*

Proof: Let $X \in B(L^2(T))$ satisfies the operator equations $\mu M_z^* X - X M_z^k = \lambda X$ as well as $\beta M_z^* X - X M_z^k = \alpha X$, that is X is generalized (λ, μ) -slant Hankel operator as well as generalized (α, β) -slant Hankel operator. Subtracting the above equation, we have $(\mu - \beta) M_z^* X = (\lambda - \alpha) X$. Substituting $\delta = (\mu - \beta)$ and $\gamma = (\lambda - \alpha)$, the preceding relation reduces to $(M_z^* - \frac{\gamma}{\delta} I) X = 0$. Now, by employing the routine observation as done in the article²⁶, we conclude that $X = 0$. Thus, the result follows.

Next, we focus our attention on certain properties (namely, linear space structure and closedness) of the class (λ, μ) -SHank_k(L^2) of all generalized (λ, μ) -slant Hankel operators on $L^2(T)$.

Theorem 2.10. *The class (λ, μ) -SHank_k(L^2) is a uniformly closed linear subspace of $B(L^2(T))$, the set of all bounded operators on $L^2(T)$.*

Proof. Let T and S be any two operators from the class (λ, μ) -SHank_k(L^2). Then, $\mu M_z^* T - T M_z^k = \lambda T$ and $\mu M_z^* S - S M_z^k = \lambda S$. For any two arbitrary

constants $\alpha, \beta \in \mathbb{C}$, take the linear combination $(\alpha T + \beta S)$ of operators T and S . It follows that from the operator $\mu M_z^*(\alpha T + \beta S) - (\alpha T + \beta S)M_z^k$, we have

$$\begin{aligned} \mu M_z^*(\alpha T + \beta S) - (\alpha T + \beta S)M_z^k &= \alpha (\mu M_z^*T - TM_z^k) + \beta (\mu M_z^*S - SM_z^k) \\ &= \lambda (\alpha T + \beta S): \end{aligned}$$

It clearly shows that $(\alpha T + \beta S) \in (\lambda, \mu)$ - $\text{SHank}_k(L^2)$ and therefore the class (λ, μ) - $\text{SHank}_k(L^2)$ is a subspace of $B(L^2(T))$.

Suppose that $(T_m)_{m \geq 1}$ is a sequence in the class (λ, μ) - $\text{SHank}_k(L^2)$ having limiting operator T , that is, $\|T_m - T\| \rightarrow 0$. Next, the expression

$$\begin{aligned} &\|(\mu M_z^*T_m - T_m M_z^k) - (\mu M_z^*T - TM_z^k)\| \\ &= \|(\mu M_z^*(T_m - T) - (T_m - T)M_z^k)\| \\ &\leq |\mu| \|M_z^*\| \|T_m - T\| + \|(T_m - T)\| \|M_z^k\|, \end{aligned}$$

converges to 0 as M_z^* and M_z^k are bounded operators. Thus, it follows that the operator $(\mu M_z^*T_m - T_m M_z^k)$ converges uniformly to $(\mu M_z^*T - TM_z^k)$ in $B(L^2(T))$. Since T_m is a generalized (λ, μ) - slant Hankel operator. Therefore, we get that

$$\lambda T_m \rightarrow (\mu M_z^*T - TM_z^k).$$

Moreover, we also have that T_m converges uniformly to T in $B(L^2(T))$. Thus, it follows that $\mu M_z^*T - TM_z^k = \lambda T$, by the uniqueness of limit, which readily yields that T is a generalized (λ, μ) - slant Hankel operator. Thus the result follows.

The following result shows the linkage amongst the elements of the class (λ, μ) - $\text{SHank}_k(L^2)$ for different unimodular complex numbers λ and μ .

Theorem 2.11. *Let α, β, λ , and μ be complex numbers with unit modulus. Then, the operator S given by*

$$S = D \frac{1}{\overline{\alpha\mu\beta\lambda}} T D \frac{1}{k\sqrt{\alpha\lambda}}$$

is a generalized (α, β) - slant Hankel operator, for each $T \in (\lambda, \mu)$ - $\text{SHank}_k(L^2)$. Here, the diagonal operator D_a is defined by $D_a(e_n) = a^n e_n$ for each integer n .

Proof: Utilizing basic properties of diagonal and multiplication operators, a routine computation gives that $D_{\frac{1}{a}} M_z^*(e_n) = a M_z^* D_{\frac{1}{a}}(e_n)$ and $M_{z^k} D_{\frac{1}{a}}(e_n) = \frac{1}{a^n} e_{n+k} = a^k \frac{1}{a^{n+k}} e_{n+k} = a^k D_{\frac{1}{a}}(e_{n+k}) = a^k D_{\frac{1}{a}} M_{z^k}(e_n)$. It readily provides that

$$D_{\frac{1}{a}} M_z^* = a M_z^* D_{\frac{1}{a}} \text{ and } M_{z^k} D_{\frac{1}{a}} = a^k D_{\frac{1}{a}} M_{z^k}.$$

With the aid of preceding observation, we get

$$D_{\frac{1}{\alpha\mu\beta\lambda}}(\mu M_z^*T - TM_{z^k})D_{\frac{1}{k\sqrt{\alpha\lambda}}} = D_{\frac{1}{\alpha\mu\beta\lambda}}(\mu M_z^*T)D_{\frac{1}{k\sqrt{\alpha\lambda}}} - D_{\frac{1}{\alpha\mu\beta\lambda}}(TM_{z^k})D_{\frac{1}{k\sqrt{\alpha\lambda}}} \\ = \bar{\alpha}\lambda\{\beta M_z^*D_{\frac{1}{\alpha\mu\beta\lambda}}(T)D_{\frac{1}{k\sqrt{\alpha\lambda}}} - D_{\frac{1}{\alpha\mu\beta\lambda}}(T)D_{\frac{1}{k\sqrt{\alpha\lambda}}}M_{z^k}\}.$$

We know that $\mu M_z^*T - TM_{z^k} = \lambda T$. Multiplying preceding relation by the operators $D_{\frac{1}{\alpha\mu\beta\lambda}}$ and $D_{\frac{1}{k\sqrt{\alpha\lambda}}}$ in the order of pre and post respectively, we have

$$\bar{\alpha}\lambda(\beta M_z^*S - SM_{z^k}) = \lambda S, \text{ where } S = D_{\frac{1}{\alpha\mu\beta\lambda}}TD_{\frac{1}{k\sqrt{\alpha\lambda}}}.$$

This immediately gives that $(\beta M_z^*S - SM_{z^k}) = \alpha S$, which implies that S is a generalized (α, β) - slant Hankel operator.

Now, we are going to investigate invariant subspaces of multiplication operator having specific inducing function in terms of generalized (λ, μ) - slant Hankel operator.

Theorem 2.12. *Let T be an operator from the space (λ, μ) - $SHank_k(L^2)$. Then the operators M_{z^k} and M_z^* have kernel $\ker(T)$ of T and closure $\overline{Ran(T)}$ of range of T as invariant subspaces, respectively.*

Proof: Assume that $T \in (\lambda, \mu)$ - $SHank_k(L^2)$. Then, we have $\mu M_z^*T - TM_{z^k} = \lambda T$, which is written in the form of $(\mu M_z^* - \lambda I)T = TM_{z^k}$. Let us assume that f be a function in the kernel $\ker T$ of T , then we have $TM_{z^k}(f) = (\mu M_z^* - \lambda I)T(f) = 0$.

This allows us to conclude that $M_{z^k}(f)$ is again in the kernel $\ker T$ of T . Thus, we get that the kernel $\ker T$ of T is an invariant subspace of M_{z^k} .

For the next part, again assume that $T \in (\lambda, \mu)$ - $SHank_k(L^2)$. then, we have $\mu M_z^*T - TM_{z^k} = \lambda T$. Consequently, it follows that

$$\bar{\mu}T^*M_z - M_{z^k}^*T^* = \bar{\lambda}T^*,$$

on taking adjoint of above relation. After re-arranging term, it has the form as $\bar{\mu}T^*M_z = (M_{z^k}^* + \bar{\lambda}I)T^*$. In the similar way as done in the above part, we obtain that $M_z(\ker T^*) \subseteq \ker T^*$. Employing the fact that $\ker(T^*)^\perp = \overline{Ran(T)}$, it immediately gives that $M_z^*(\overline{Ran(T)}) \subseteq \overline{Ran(T)}$. Thus the result follows.

The next theorem furnishes a result for a generalized (λ, μ) - slant Hankel operator, which is quite similar to the much-celebrated classical theorem for Hankel operator by Kronecker. In the discussion of following result, the boundedness condition on the operator T is relaxed for sake of convenience.

Theorem 2.13. *Let T be an operator from the space (λ, μ) - $SHank_k(L^2)$. Then, the necessary and sufficient condition under which the matrix of T , with respect to orthonormal basis $\{e_n\}$ of $L^2(T)$, is of finite rank matrix is that each of the function $T(e_i)$ is rational function for all $i \in \{0, 1, 2, \dots, k-1\}$.*

Proof: Let $T \in (\lambda, \mu)$ - SHank $_k(L^2)$, it satisfies the relation $\mu M_z^* T - T M_{z^k} = \lambda T$. Equivalently, we have $(\mu M_z^* - \lambda I)T = T M_{z^k}$. For $i \in \{0, 1, 2, \dots, k-1\}$, the repeated application of the preceding relation yields that

$$\begin{aligned}
 (2.6) \quad T(e_{km+i}) &= T M_{z^k}^m(e_i) \\
 &= (\mu M_z^* - \lambda I) T M_{z^k}^{m-1}(e_i) \\
 &= (\mu M_z^* - \lambda I)^2 T M_{z^k}^{m-2}(e_i) \\
 &= (\mu M_z^* - \lambda I)^m T(e_i)
 \end{aligned}$$

It is obvious from the relation (2.6) that the operator T can be devised utilizing the function $T(e_i) = \varphi_i$ in a unique way for each $i \in \{0, 1, 2, \dots, k-1\}$. In the view of the relation (2.6), the inner product $\langle T(e_{km+i}) | e_n \rangle$ reduces to

$$\begin{aligned}
 (2.7) \quad \langle T(e_{km+i}) | e_n \rangle &= \langle (\mu M_z^* - \lambda I)^m T(e_i) | e_n \rangle \\
 &= \left\langle T(e_i) | (\bar{\mu} M_z - \bar{\lambda} I)^m e_n \right\rangle \\
 &= \sum_{k=0}^m \binom{m}{k} (-\lambda)^{m-k} \mu^k \langle T(e_i) | M_z^k e_n \rangle \\
 &= \sum_{k=0}^m \binom{m}{k} (-\lambda)^{m-k} \mu^k \langle T(e_i) | e_{n+k} \rangle
 \end{aligned}$$

for each $i \in \{0, 1, 2, \dots, k-1\}$. The outputs obtained in the relations (2.6) and (2.7) are enough to conclude that each k consecutive columns of the matrix of T with respect to orthonormal basis are related to the functions $(\mu M_z^* - \lambda I)^m T(e_i)$ as the corresponding co-ordinate vector in certain order. Let $n_0 = \sum_{i=0}^{k-1} N_i$ be the rank of the matrix of T , which is finite. Then there exists $n_0 + 1$ constants (namely, $a_{p_0}^0, 0 \leq p_0 \leq N_0$ and for $\neq 0, a_{p_i}^i, 0 \leq p_i \leq N_i - 1$), not all zero such that

$$(2.8) \quad \sum_{p_0=0}^{N_0} a_{p_0}^0 (\mu M_z^* - \lambda I)^{p_0} \varphi_0 + \sum_{i=1}^{k-1} \sum_{p_i=0}^{N_i-1} a_{p_i}^i (\mu M_z^* - \lambda I)^{p_i} \varphi_i = 0.$$

Again, by a routine computation, we have

$$\begin{aligned}
 (2.9) \quad \sum_{p_0=0}^{N_0} a_{p_0}^0 (\mu M_z^* - \lambda I)^{p_0} \varphi_0 &= \sum_{p_0=0}^{N_0} a_{p_0}^0 \sum_{k_0=0}^{p_0} \binom{p_0}{k_0} (-\lambda)^{p_0-k_0} \mu^{k_0} M_z^{*k_0} \varphi_0 \\
 &= \sum_{k_0=0}^{N_0} \sum_{p_0=k_0}^{N_0} a_{p_0}^0 \binom{p_0}{k_0} (-\lambda)^{p_0-k_0} \mu^{k_0} M_z^{*k_0} \varphi_0.
 \end{aligned}$$

For $i \in \{0, 1, 2, \dots, k-1\}$, applying the similar technique as done in the relation (2.9), we get that

$$\begin{aligned}
 (2.10) \quad \sum_{p_i=0}^{N_i-1} a_{p_i}^i (\mu M_z^* - \lambda I)^{p_i} \varphi_i &= \sum_{p_i=0}^{N_i-1} a_{p_i}^i \sum_{k_i=0}^{p_i} \binom{p_i}{k_i} (-\lambda)^{p_i-k_i} \mu^{k_i} M_z^{*k_i} \varphi_i \\
 &= \sum_{k_i=0}^{N_i-1} \sum_{p_i=k_i}^{N_i-1} a_{p_i}^i \binom{p_i}{k_i} (-\lambda)^{p_i-k_i} \mu^{k_i} M_z^{*k_i} \varphi_i.
 \end{aligned}$$

With the help of relations (2.9) and (2.10), the expression appearing on the left-hand side of the equation (2.8) can be written as

$$\begin{aligned}
(2.11) \quad & \sum_{p_0=0}^{N_0} a_{p_0}^0 (\mu M_z^* - \lambda I)^{p_0} \varphi_0 + \sum_{i=1}^{k-1} \sum_{p_i=0}^{N_i-1} a_{p_i}^i (\mu M_z^* - \lambda I)^{p_i} \varphi_i \\
&= \sum_{k_0=0}^{N_0} \sum_{p_0=k_0}^{N_0} a_{p_0}^0 \binom{p_0}{k_0} (-\lambda)^{p_0-k_0} \mu^{k_0} M_z^{*k_0} \varphi_0 + \\
&\quad + \sum_{i=1}^{k-1} \sum_{k_i=0}^{N_i-1} \sum_{p_i=k_i}^{N_i-1} a_{p_i}^i \binom{p_i}{k_i} (-\lambda)^{p_i-k_i} \mu^{k_i} M_z^{*k_i} \varphi_i \\
&= \sum_{k_0=0}^{N_0} c_{k_0}^0 M_z^{*k_0} \varphi_0 + \sum_{i=1}^{k-1} \sum_{k_i=0}^{N_i-1} c_{k_i}^i M_z^{*k_i} \varphi_i.
\end{aligned}$$

The constants $c_{k_0}^0$ and $c_{k_i}^i$, for $i \in \{1, 2, \dots, k-1\}$, appearing in the above relation are given by

$$\begin{aligned}
c_{k_0}^0 &= \sum_{p_0=k_0}^{N_0} a_{p_0}^0 \binom{p_0}{k_0} (-\lambda)^{p_0-k_0} \mu^{k_0} \\
c_{k_i}^i &= \sum_{p_i=k_i}^{N_i-1} a_{p_i}^i \binom{p_i}{k_i} (-\lambda)^{p_i-k_i} \mu^{k_i}.
\end{aligned}$$

For $i \in \{1, 2, \dots, k-1\}$, the relations (2.8) and (2.11) allows us to conclude that $c_{k_0}^0 = c_{k_i}^i = 0$ for $0 \leq p_0 \leq N_0$ and $0 \leq k_i \leq N_i - 1$ if and only if $a_{p_0}^0 = a_{p_i}^i = 0$ for $0 \leq p_0 \leq N_0$ and $0 \leq p_i \leq N_i - 1$. Equivalently, for $i \in \{1, 2, \dots, k-1\}$, we derive that the set

$$S = \{(\mu M_z^* - \lambda I)^{p_0} \varphi_0, (\mu M_z^* - \lambda I)^{p_i} \varphi_i : 0 \leq p_0 \leq N_0, 0 \leq p_i \leq N_i - 1\}$$

is a linearly independent set if and only if the set S_0 given by $S_0 = \{M_z^{*p_0} \varphi_0, M_z^{*p_i} \varphi_i : 0 \leq p_0 \leq N_0, 0 \leq p_i \leq N_i - 1\}$ is also linearly independent set. From the above observations, it is easy to see that the function $\varphi = \sum_{i=0}^k \varphi_i$ acts as a inducing symbol of the Hankel operator, whose matrix representation have the co-ordinate vectors corresponding to $M_z^{*p_0} \varphi_0$ and $M_z^{*p_i} \varphi_i$ as its columns, where $0 \leq p_0 \leq N_0$ and $0 \leq p_i \leq N_i - 1$. Therefore, the Hankel operator with symbol $\varphi = \sum_{i=0}^k \varphi_i$ is an operator of finite rank if and only if the operator T also have finite rank. With the aid of much celebrated Kronecker theorem related with Hankel operator, we find that all the functions φ_i , for $i \in \{0, 1, 2, \dots, k-1\}$, are rational functions.

Let us revisit to our main goal of analysis, which consists of investigation of non-trivial solutions of the operator equation $\mu M_z^* T - T M_z^k = \lambda T$ for an arbitrarily choose complex numbers λ and μ .

Theorem 2.14. *The operator equation $\mu M_z^* T - T M_z^k = \lambda T$ admits non-zero solution under the condition $|\mu| \neq 0$, and $|\lambda| < 1 + |\mu|$ and trivial solutions under the following condition:*

- (1) $|\mu| < 1$ and $|\lambda| > 1 + |\mu|$;
- (2) $|\mu| > 1$ and $|\lambda| > 1 + |\mu|$.

Proof: It is well known that $\mu\omega - \lambda \in D$ if and only if $|\lambda| < 1 + |\mu|$ for complex numbers $\mu \neq 0, \lambda$ and $\omega \in D$, where D is the open unit disk. Therefore, at the first glance, it is easy to find a number ω in open unit disk D such that

$\mu\omega - \lambda$ is also in D. It also allows us to choose function S_ω , defined by (2.1). Utilizing the function S_ω , let us define a rank one operator T given by

$$T = S_{\bar{\omega}} \otimes S_{k\sqrt{\mu\omega - \lambda}}.$$

Note that $R(f \otimes h)S = (Rf) \otimes (S^*h)$ for all operators R and S from $B(L^2(T))$ and $f, h \in L^2(T)$. Now, consider the operator $\mu M_z^* T - TM_{z^k}$, which gives that

$$\begin{aligned} (2.12) \quad \mu M_z^* T - TM_{z^k} &= \mu M_z^* (S_{\bar{\omega}} \otimes S_{k\sqrt{\mu\omega - \lambda}}) - (S_{\bar{\omega}} \otimes S_{k\sqrt{\mu\omega - \lambda}}) M_{z^k} \\ &= \mu (M_z^* S_{\bar{\omega}} \otimes S_{k\sqrt{\mu\omega - \lambda}}) - (S_{\bar{\omega}} \otimes M_{z^k}^* S_{k\sqrt{\mu\omega - \lambda}}). \end{aligned}$$

A routine computation derives the fact that $M_z^* S_{\bar{\omega}} = \omega S_{\bar{\omega}}$ and $M_{z^k}^* S_{\bar{\omega}} = \omega^k S_{\bar{\omega}}$ for each ω in open unit disk D. Employing the above relation, the equation (2.12) reduces to

$$\begin{aligned} \mu M_z^* T - TM_{z^k} &= \mu\omega (S_{\bar{\omega}} \otimes S_{k\sqrt{\mu\omega - \lambda}}) - (\mu\omega - \lambda)(S_{\bar{\omega}} \otimes S_{k\sqrt{\mu\omega - \lambda}}) \\ &= (\lambda)(S_{\bar{\omega}} \otimes S_{k\sqrt{\mu\omega - \lambda}}) = \lambda T. \end{aligned}$$

It clearly provides a non-zero solution $T = S_{\bar{\omega}} \otimes S_{k\sqrt{\mu\omega - \lambda}}$ of the equation $\mu M_z^* T - TM_{z^k} = \lambda T$ under the condition $|\mu| \neq 0$, $|\lambda| < 1 + |\mu|$. Particularly, it is a nontrivial solution of the considered operator equation for complex numbers λ, μ satisfying $0 < |\mu| < 1$, $|\lambda| < 2|\mu|$.

Now, we analyze the cases under which the equation $\mu M_z^* T - TM_{z^k} = \lambda T$ has only trivial solution.

Case: 1 When $|\mu| < 1$ and $|\lambda| > 1 + |\mu|$. In this case, define an operator R_μ on $B(L^2(T))$ given by

$$R_\mu(X) = \mu M_z^* X - X M_{z^k},$$

for all $X \in B(L^2(T))$. From the part first, one may derive that the operator R_μ has a complex number λ as its eigenvalue if λ satisfies $|\lambda| < 2|\mu|$. Consequently, it derives that the set $\{z \in \mathbb{C} : |\lambda| < 2|\mu|\}$ is subset of the spectrum $\sigma(R_\mu) \subseteq \{z \in \mathbb{C} : |\lambda| < 1 + |\mu|\}$ of R_μ . Equivalently, we obtain that $|\lambda| > 1 + |\mu|$ is not a contained in $\sigma(R_\mu)$. Therefore equation $\mu M_z^* T - TM_{z^k} = \lambda T$ admits only null operator as its solution.

Case: 2 When $|\mu| > 1$ and $|\lambda| > 1 + |\mu|$. In this case, again, define an operator $R_{\frac{1}{\mu}}: B(L^2(T)) \rightarrow B(L^2(T))$ given by $R_{\frac{1}{\mu}}(Z) = M_z^* Z - \frac{1}{\mu} Z M_{z^k}$ for all operator Z in the space $B(L^2(T))$. As we know that for any operator T, $\sigma(T) \subseteq \{z \in \mathbb{C} : |z| \leq \|T\|\}$. Therefore, we have

$$\sigma\left(R_{\frac{1}{\mu}}\right) \subseteq \left\{z \in \mathbb{C} : |z| \leq \left\|R_{\frac{1}{\mu}}\right\|\right\} \subseteq \left\{z \in \mathbb{C} : |z| \leq 1 + \frac{1}{|\mu|}\right\}.$$

As we have assumed that $|\lambda| > 1 + |\mu|$, equivalently

$$\left| \frac{\lambda}{\mu} \right| > 1 + \frac{1}{|\mu|}.$$

This provides $R_{\frac{1}{\mu}} - \frac{\lambda}{\mu}I$ as an invertible operator. Thus, we can infer that there is only null operator, which is solution of operator equation $\mu M_z^* Z - Z M_{z^k} = \lambda Z$. This completes the proof.

Employing the Theorem 2.14 and the same technique²², the following result can be easily obtained.

Corollary 2.15. *There is only zero operator, which acts as a solution of the equation $\mu M_z^* X - X M_{z^k} = \lambda X$ for the complex numbers λ, μ satisfying the relation $|\mu| < 1, |\lambda| \geq 2$. Moreover, the same result is also true under the condition $|\mu| > 1, |\lambda| \geq 2|\mu|$.*

From the Theorem 2.14, we devised a result, which also furnishes non-trivial solution for the equation

$$(2.13) \quad \mu M_z^* X - X M_{z^k} = \lambda X$$

Corollary 2.16. *The operator equation, given in (2.13), admits non-zero operator as its solution if constants λ, μ satisfies one of the following conditions.*

$$(1) \quad |\mu| < 1 \text{ and } |\lambda| \leq 2|\mu|$$

$$(2) \quad |\mu| > 1 \text{ and } |\lambda| \leq 2$$

Proof: For a complex number μ satisfying $|\mu| < 1$, the relation $|\lambda| \leq 2|\mu|$ yields $|\lambda| \leq 2|\mu| < 1 + |\mu|$. Similarly, for a complex number μ with the condition $|\mu| > 1$, the inequality $|\lambda| \leq 2$ provides that $|\lambda| \leq 2 < 1 + |\mu|$. Now, with the aid of the Theorem 2.14, the result follows trivially.

3. Conclusion

The problems (namely, to solve $U^* X U = \lambda X$ and $M_z X - X M_{z^2} = 0$), proposed by Barria, Halmos² and M.C. Ho¹⁰ respectively, gave a new insight to consider different generalizations of Hankel and slant Hankel operators. It motivate Avendano⁴, Sun⁸, Datt, Aggarwal¹³ and other mathematician to consider different forms of Hankel operator (namely, λ - Hankel operator⁴; (λ, μ) - Hankel operator, λ - Slant Hankel operator^{13,14,19}, (λ, μ) - slant Hankel operator²⁶, λ - Slant Hankel operator via spectral problem²⁷ respectively). This paper generalizes the notion of slant Hankel operator via operator equation $\mu M_z^* X - X M_{z^k} = \lambda X$. Here, we solve the equation $\mu M_z^* X - X M_{z^k} = \lambda X$ completely and discuss certain specific properties of its solution, which is called as generalized (λ, μ) - slant Hankel operator. Our study reduces to several well-known results and facts (investigated by

mathematician of this area of research) for the particular choice to the constants k, λ, μ . For instance, if we take $\lambda = 0, \mu = 1$ and $k = 1$, then our study reduces to the study of Hankel operator on $L^2(T)$ and if we take $k = 2, \lambda = 0$ and $\mu = 1$, then our study give results related to slant Hankel operator done by Arora and Batra^{1,16}.

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