

Reliability Analysis of A New Life Testing Model

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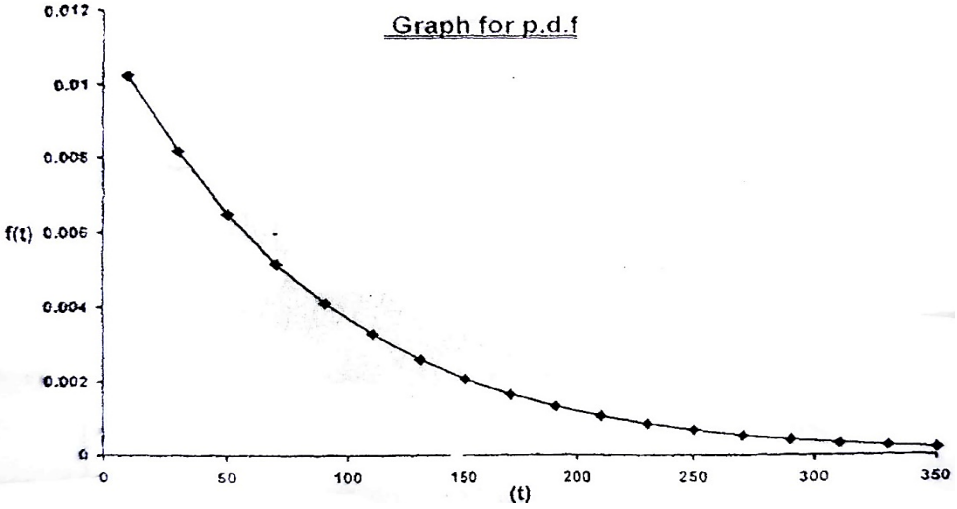
Abstract: In Present time, the utility of the theory of reliability and lifetesting analysis attracting many workers and theory is well established in engineering and medical sciences. Now, as the area of applicability of life testing is enlarging day by day, the variety of situations and their related data are also growing up. So, it seems not logical that this variety of data can be dealt with only two or three types of the above life testing models. It seems desirable to develop more and more realistic life testing models to deal with the variety of life testing data so that the proper inferences concerning with the reliability and other aspects can be drawn with more precision. It has been observed that in a number of situations the application of the models discussed so far can be made only after imposing a number of assumptions. So, there is a requirement of a relatively larger family of life time models, so that proper justification can be done with the daya analysis. In the present paper, an attempt has been made to develop a new life-testing model to enlarge the family of life testing models.

Keywords: Reliability, Lifetesting Model, Failure rate, Hazard rate, Probability density, function, distributions.

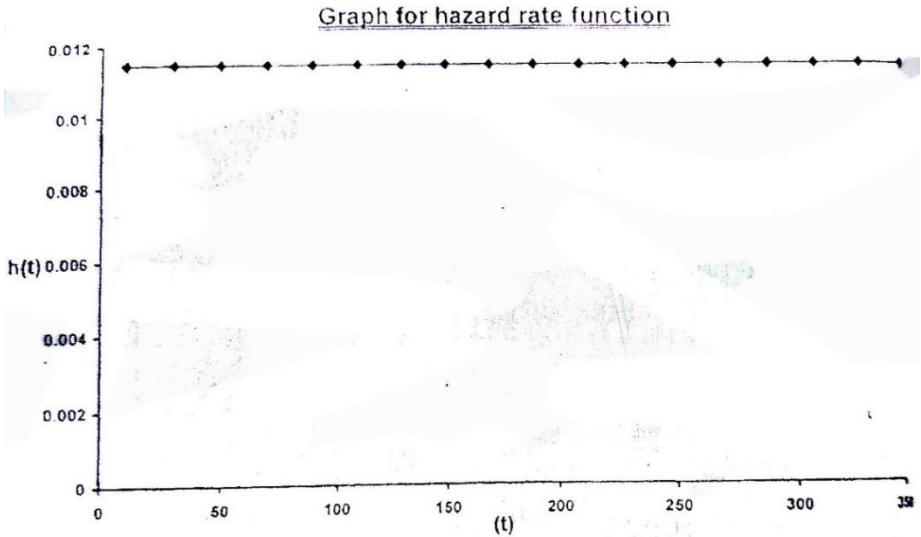
1. Introduction

The parametric modelling is the important tool for any scientific analysis. In early days some special type of probability distributions were being used as life testing model and still they are in practice, actually these probability distributions possesses some special features required to be in the models to use them as life testing models, for example they should be mirror J type in shape so that the concept of reducing probability could be justified and they must lie in positive quadrant etc. Initially the Exponential distribution was in use and a lot of early work is found through this model but in 1951 Weibull established a new model, which is known after this name.

In 1960's, a number of workers have been done a lot of efforts to enlarge the family of life testing models but all those models have been related either to exponential or to geometric distribution. Later on, some workers characterised well-known distribution viz. Bionomial, Poisson, mixture of exponential distribution and made an attempt to justify there utility as life testing model/distributions.



Graph 1



Graph 2

The model has been widely discussed with its characterization and some important properties. A real life data from Davis (1952) have been used to

make numerical illustration of the proposed model to prove its practical applicability in real life situations. The proposed model is a classic inverse J-Shaped and with constant failure rate [(Graph 1 and Graph 2].

2. Model Development

The demanding situation is to develop a model considering the randomness of the situation as in the deterministic modelling the randomness of the factors is not considered it fixes some parameters affecting the failures. Through the phenomenon of failure is being studied either in any form.

In the probabilistic modelling approach, there exist three methodologies to study the same in the available literature so far. These three methodologies can be summarized as follows:

Through Existing Models: In this approach the existing probability density functions in the available literature are studied and characterized to establish them as life time models. This approach was initiated by Weibull (1951), when he established his own probability density function as life time model. Later on, in 1954, Epstein and Sobel studied and characterized the well known exponential probability density function as life time model. Also, Folks and Chhikara (1978) worked on inverted gamma probability density function and justified its applicability in life testing theory.

Through Models Developed For Particular Situations: The second approach consists of the development of probabilistic models for lifetime data based on one's requirement. In this approach a mathematical expression is being framed in such a manner that it satisfies the limits of the variable and parameters under study and also satisfies the requirement of being a probability density function. Using the same approach, Mukherjee-Islam (1983) and Siddiqui et al. (1992, 1995) developed some life time models to study some particular types of life time data.

Through Mathematical Derivation of Models: This approach is based on the role of hazard rate function in life testing analysis and the fundamentals of the subject. This technique is not commonly known so far. But still a number of workers have done some work using the same approach. An extensive discussion on the hazard rate function in Cox (1962) straightened the idea of this approach. Krane (1963) and Koldin (1966) considered polynomial of various degrees as hazard rate function and developed the life time models based on them. Also, Mohn et al. (1970) used another expression to reformulate the well known Gompertz (1825) distribution. Gupta (1994) has developed a classic inverse J-shaped model by considering the situations where the hazard rate function may be parabolic shape.

The above technique is a direct implementation of the fact that if the hazard rate for any data is known in the form of a numerical equation, the lifetime model can uniquely be determined on the basis of given hazard rate. The procedure has been discussed in detail in the following section.

3. Procedure

Consider an unknown differentiable function $R(t)$, having the capacity to define the proportions of survivors out of the total population exposed to risk. Call this function as reliability function.

Check the function considered as a reliability function, i.e. whether it satisfies the following conditions or not.

$$(1) \quad R(0) = 1; R(\infty) = 0 \quad \text{and} \quad R'(t) \leq 0 \text{ for all } t \geq 0.$$

If it satisfies the above condition then the corresponding probability density function can be obtained as;

$$(2) \quad f(t) = -d[R(t)]$$

and the corresponding hazard rate function which is defined as;

$$h(t) = f(t)/R(t)$$

Equating the value of $f(t)$ from (2) and (3),

$$(3) \quad h(t) = [-dR(t)]/R(t)$$

$$(4) \quad h(t) = -d [\log R(t)]$$

In order to obtain $R(t)$ and equation (4) can be reformulated as follows

$$(5) \quad R(t) = \exp \left(- \int_0^t h(u) du \right) = \exp[-h(t)]$$

Putting value of $R(t)$ from (5) to (2), finally the following result can be obtained;

$$f(t) = h(t) \exp(- \int h(u) du)$$

The equation (4) is exponentially distributed with mean unity (Krane; 1963). Hence, the $h(t)$ obtained in (5) should bound under the following conditions;

$$(6) \quad h(0) = 0; h(\infty) = \infty; \quad \text{and}$$

$$(7) \quad h'(t) \geq 0 \text{ for all } t \geq 0$$

So, it can be stated in the light of above discussion and equation (6) and (7) that if the hazard rate $h(t)$ is known, the failure model or probability density function $f(t)$ can be uniquely be defined.

Keeping the same concept of model development for life testing analysis, an attempt has been made in the following section. The hazard rate considered here for the purpose is constant. While working with the data provided by

Sinha (1952), it seems that a modification can be made on the hazard rate function, so that the model developed for that hazard rate function represents well the life time data with more precision. So that the inferences drawn from the model on the basis of data may be of greater accuracy. In some situations, where the exponential model with constant hazard rate equals to the parameter say ' θ ' seems to be fitted well, it has been observed that if the hazard rate function is taken as $\log_e \theta$ then the model developed for this hazard rate will be more appropriate to study the life time data.

4. Proposed Infinite Range Failure Model

The proposed model has the following p.d.f.

$$f(x) = \theta^{-x} \log_e \theta \quad ; x, \theta \geq 0$$

The cumulative distribution function [Graph (3)] is as follows;

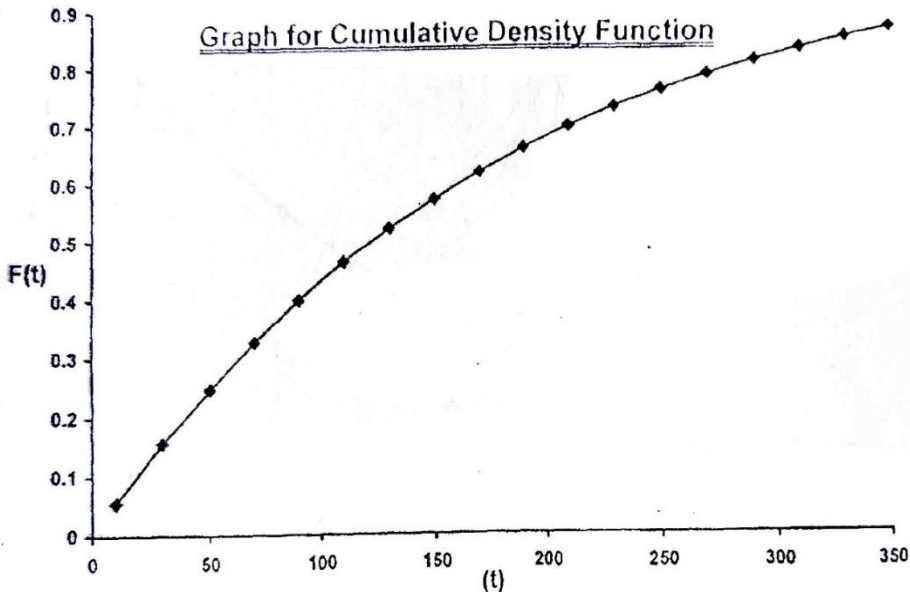
$$F(x) = 1 - \theta^{-x}$$

The reliability function [Graph (4)] and hazard rate [Graph (2)] function are

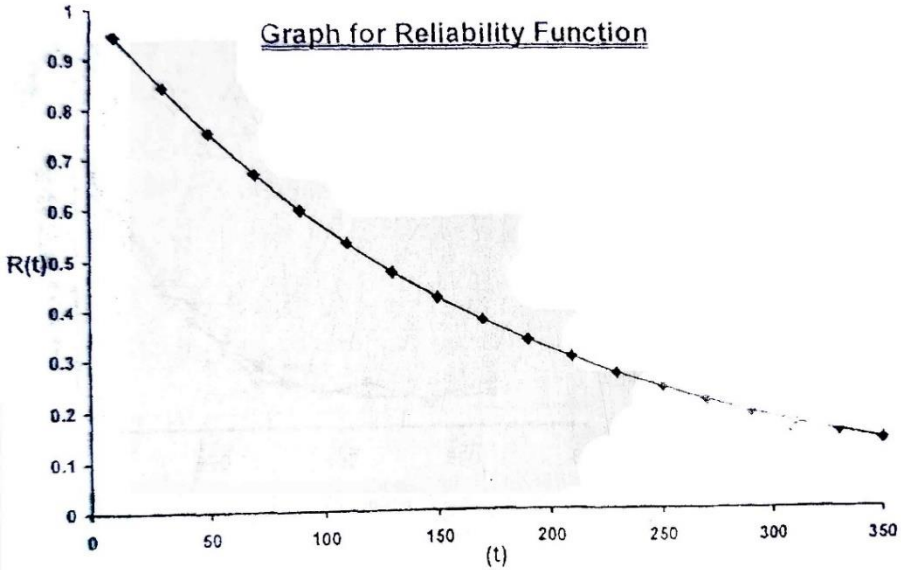
$$R(x) = \theta^{-x}$$

and

$$h(x) = \log_e \theta$$



Graph 3



Graph 4

Characterization of the Proposed Model

Following are the main constant of the proposed model which help in characterizing the same.

(1) Mean

$$E(x) = \int_0^\infty x \cdot f(x) dx$$

so, Mean = $1/\log_e \theta$

Similarly, we can obtain

$$E(x^2) = \int_0^\infty x^2 \cdot f(x) dx$$

$$E(x^2) = 2/(\log_e \theta)^2$$

(2) Variance

Now the variance can be obtained by using the expression .

$$V(x) = E(x^2) - [E(x)]^2$$

so, Variance = $1/(\log_e \theta)^2$

(3) Co-Efficient For Variation

$$C.V. = [\text{Standard deviation}/\text{mean}]$$

$$= 1$$

The coefficient of variation is pure constant and independent of the parameter θ . Such an expression for CV is not seen in other lifetime models, and its utility is yet to be explored.

(4) Median

$$\int_0^{-Me} f(x) dx = 1/2$$

which yields the value of median as;

$$\text{Median} = [\log_e(1/2)] / \log_e \theta$$

(5) Moment Generating Function

The m.g.f. about origin is as follows;

$$M_x(t) = \int_x e^{tx} \cdot f(x) dx$$

$$\text{so, mgf} = \log_e \theta [\log_e(\theta/e^{it})]^{-1}$$

(6) Characteristic Function

The characteristic function of the proposed model has obtained as follows;

$$E[e^{itx}] = \int_x e^{itx} \cdot f(x) dx = \log_e \theta [\log_e(\theta/e^{it})]^{-1}$$

(7) Forgetfulness Property

The most desirable property among the life time models is the property of forgetfulness, specially when analysis of electronic items is to be done.

The proposed model follows forgetfulness property; the statement can be justified by the following proof.

$$P[Y \leq x | X \geq a] = P[X \geq a]$$

where $Y = X - a$

Now,

$$\begin{aligned} P[Y \leq x \cap X \geq a] &= P[X - a \leq x \cap X \geq a] \\ &= P[X \leq x + a \cap X \geq a] \\ &= P[a \leq X \leq X + a] \\ &= \int_a^{x+a} f(x) dx \\ &= \theta^{-a}(1 - \theta^{-x}) \end{aligned}$$

$$P[X \leq x] = 1 - \theta^{-x}$$

and

$$P[X \geq a] = \theta^{-a}$$

Now

$$\begin{aligned} P[Y \leq x | X \geq a] &= P[Y \leq x \cap X \geq a] / P[X \geq a] \\ &= 1 - \theta^{-x} \\ &= P[X \leq x] \end{aligned}$$

Hence, we can state that the proposed model possesses the forgetfulness property.

(8) Maximum Likelihood Estimates for Parameter

In order to obtain m.l.e of the parameter ' θ ', the likelihood function will be as;

$$L = \prod_{i=1}^n f(X_i, \theta) = (\log_e \theta)^n \theta^{-\sum x_i}$$

Taking log on both side and differentiating partially with respect to ' θ ' and equating to zero; the m.l.e. of the parameter ' θ ' can be obtain as;

$$\hat{\theta} = \exp(1/\bar{x}).$$

(9) Average Failure Rate

The average failue rate over the interval (0,t) can be obtained as

$$AFR(t) = Ln\theta$$

and the same over the interval (x_1x_2) will be

$$AFR_{(x_1x_2)} = -\ln(\theta^{-x_1} - \theta^{-x_2})/(x_2 - x_1)$$

(10) Series and Parallel Systems Lives

The two expressions for the distribution of $Y_1 = \min(x_1x_2)$ and $Y_2 = \max(x_1x_2)$ can be derived in order to represent the series and parallel system lives. It is assumed here that x_1 and x_2 are identically and independently distributed.

Therefore

$$\begin{aligned} f(y_1) &= 2. \theta^{-2x} \log_e \theta \\ f(y_2) &= 2. \theta^{-x} \log_e \theta (1-\theta^{-x}) \end{aligned}$$

5. Numerical Illustration

The data used to illustrate the applicability of the proposed model is given in the following table provided by Davis (1952)

Time Interval	Observed Frequency	Expected Frequency
0-50	82	89
50-100	75	67
100-150	49	51
150-200	45	38
200-250	27	29
250-300	17	22
300-350	16	16
350-400	13	12
400-500	11	16
500-600	10	9
600 & above	16	12
Total	361	361

For the given data the m.l.e of the parameter ' θ ' is

$$\hat{\theta} = 1.0057$$

and the calculated value of chi-square is

$$\chi^2_{\text{cal}} = 7.0248 \text{ with 9 degree of freedom.}$$

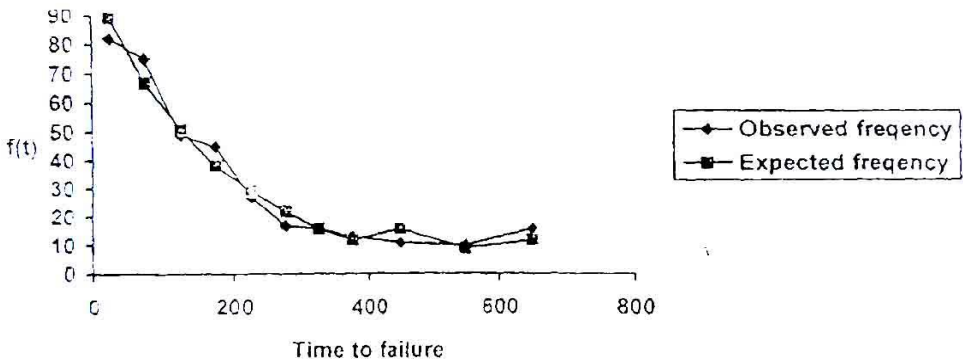
While $\chi^2_{\text{tab}} = 16.919$ at 5% level of significance.

Hence, the data seems fit well to the proposed model.

6. Conclusion

The proposed model is an effort towards increasing the class of life testing models. The discussion so far given in the previous sections clearly gives an impression that model has all desired characteristics of a good life testing model. The two interesting features are that it possess the property of new better than used and its co-efficient of variation is a pure constant. The data fitted clearly indicates the practical importance of the proposed model in the field of reliability analysis. Graph for the fitted data has been shown in Graph 5. Also the model has the property of forgetfulness, which has its own importance in certain type of specific failure analysis.

Graph for fitted data



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