

Infinitesimal Affine Motions in a Finsler Space Equipped with Cartan's Connection

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1. Introduction

Infinitesimal affine motions generated by contra, concurrent, special concircular, recurrent, concircular, torse forming and birecurrent vectors in special types of Riemannian, non-Riemannian and Finslerian spaces have been studied by various authors including K. Takano¹⁻², R.S. Sinha³, R.B. Misra⁴, R.B. Misra and F.M. Meher⁵⁻⁷, F.M. Meher⁸, A. Kumar⁹⁻¹¹, P.N. Pandey^{12-15, 16-18}, P.N. Pandey and V.J. Dwivedi¹⁹. All of the above authors except the first author¹⁶⁻¹⁸ of the present paper concentrated themselves on the problem of finding necessary consequences of the affine motion generated by above vectors in a space of recurrent curvature and some other special spaces. The basic problem of finding the necessary and sufficient conditions for the above vectors to generate an affine motion in a general Finsler space is solved only by the first author of the present paper in his papers¹⁶⁻¹⁸. The discussions on affine motions in different types of special Finsler spaces are also included there in and most of the results obtained by the above authors in this direction have been generalised. The whole discussion of the papers¹⁶⁻¹⁸ is based on Berwald's connection and thus, the discussion based on Cartan's connection is untouched. The aim of the present paper is to discuss the problem of finding the necessary and sufficient conditions for the above vectors to generate an affine motion in a Finsler space equipped with E. Cartan's connection. The notation of the present paper is based on H.Rund²⁰.

2. Preliminaries

Let $F_n(F, g, \Gamma)$ be an n -dimensional Finsler space of class atleast C^7 equipped with a metric function F^* satisfying the required conditions²⁰, the corresponding symmetric tensor g and the Cartan's connection Γ . The relations between the metric

* Unless otherwise stated, all the geometric objects are supposed to be functions of the line-elements $(x^i, \dot{x}^i)$. The indices $i, j, k \dots$ take positive integer values from 1 to n .

function F and the components g_{ij} of the corresponding metric tensor g are given by

$$(2.1) \quad (a) g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F^2, \quad (b) g_{ij} \dot{x}^i \dot{x}^j = F^2,$$

where $\dot{\partial}_i = \partial / \partial \dot{x}^i$. Putting $g_{ij} \dot{x}^j = y_i$ in (2.1) and using the fact that the metric function F is positively homogeneous of degree one in $\dot{x}^i$, we have

$$(2.2) \quad (a) g_{ij} = \dot{\partial}_i y_j, \quad (b) y_i \dot{x}^i = F^2.$$

The tensor C_{ijk} , defined as $C_{ijk} = \frac{1}{2} \dot{\partial}_k g_{ij}$, is symmetric in its all indices and satisfies

$$(2.3) \quad C_{kij} \dot{x}^k = C_{ikj} \dot{x}^k = C_{ijk} \dot{x}^k = 0.$$

Due to (2.3) and symmetry of C_{ijk} in its indices the tensor C_{jk}^i which is defined as $C_{jk}^i = g^{ip} C_{jpk}$, satisfies

$$(2.4) \quad (a) C_{jk}^i = C_{kj}^i, \quad (b) C_{jk}^i \dot{x}^k = C_{kj}^i \dot{x}^k = 0.$$

The coefficients $\dot{\Gamma}_{jk}^i$ of Cartan's connection $\dot{\Gamma}$ are symmetric in their lower indices and positively homogeneous of degree zero in $\dot{x}^i$. Due to homogeneity of $\dot{\Gamma}_{jk}^i$ in $\dot{x}^i$, the connection coefficients $\dot{\Gamma}_{jk}^i$ satisfy

$$(2.5) \quad \dot{x}^h \dot{\partial}_h \dot{\Gamma}_{jk}^i = 0.$$

The covariant derivative of an arbitrary tensor T_j^i for the connection $\dot{\Gamma}^*$ is given by

$$(2.6) \quad T_{j|k}^i = \partial_k T_j^i - (\dot{\partial}_r T_j^i) G_k^r + T_j^r \dot{\Gamma}_{rk}^i - T_r^i \dot{\Gamma}_{jk}^r,$$

where $G_k^r = \dot{\Gamma}_{kh}^r \dot{x}^h$ and $\partial_k = \partial / \partial x^k$. The commutation formula for the differential operators $\dot{\partial}_k$ and $|_h$ are given by

$$(2.7) \quad \dot{\partial}_k (X^i_{|h}) - (\dot{\partial}_k X^i)_{|h} = (\dot{\partial}_k \Gamma^i_{rh}) X^r - \dot{x}^m (\dot{\partial}_k \Gamma^r_{hm}) \dot{\partial}_r X^i,$$

$$(2.8) \quad X^i_{|hk} - X^i_{|kh} = X^r K^i_{rhk} - (\dot{\partial}_r X^i) K^r_{shk} \dot{x}^s,$$

where K^i_{jkh} constitute Cartan's curvature tensor. This tensor is skew symmetric in last two lower indices and positively homogeneous of degree zero in $\dot{x}^i$ s. The curvature tensor K^i_{jkh} is connected with the tensor H^i_{kh} , defined by Berwald, by the equation

$$(2.9) \quad K^i_{hjk} \dot{x}^h = H^i_{jk}.$$

The tensor H^i_{jk} satisfies²¹

$$(2.10) \quad y_i H^i_{jk} = 0.$$

The associate curvature tensor K_{jpkh} , defined by

$$(2.11) \quad g_{ip} K^i_{jkh} = K_{jpkh},$$

is skew symmetric in its last two indices.

Let us consider an infinitesimal transformation

$$(2.12) \quad \bar{x}^i = x^i + \epsilon v^i(x^j)$$

generated by a vector $v^i(x^j), \epsilon$ being an infinitesimal constant. Denoting the operator for Lie differentiation with respect to the above transformation by the symbol $\mathcal{L}$, the Lie derivative of a tensor T^i_j and Cartan's connection coefficients are given by

$$(2.13) \quad \mathcal{L} T^i_j = T^i_{j|r} v^r - T^r_j v^i_{|r} + T^i_r v^r_{|j} + (\dot{\partial}_r T^i_j) v^r_{|s} \dot{x}^s$$

and

$$(2.14) \quad \mathcal{L} \Gamma^i_{jk} = v^i_{|jk} + K^i_{jkh} v^h + (\dot{\partial}_h \Gamma^i_{jk}) v^h_{|s} \dot{x}^s.$$

An infinitesimal transformation²¹ becomes an affine motion if the Lie derivative of the connection coefficients Γ^i_{jk} vanishes identically, i.e.

$$(2.15) \quad \mathcal{L} \Gamma_{jk}^* = 0.$$

A vector v^i is called contra, concurrent, special concircular, recurrent, concircular, torse forming and birecurrent according as it satisfies

- (a) $v_{|k}^i = 0$,
- (b) $v_{|k}^i = C \delta_k^i$, C being a constant
- (c) $v_{|k}^i = \rho \delta_k^i$, ρ is not a constant.
- (d) $v_{|k}^i = \mu_k v^i$,
- (e) $v_{|k}^i = \mu_k v^i + \rho \delta_k^i$, $\mu_{k|j} = \mu_{j|k}$
- (f) $v_{|k}^i = \mu_k v^i + \rho \delta_k^i$,
- (g) $v_{|jk}^i = \varphi_{jk} v^i$,

respectively. The affine motion generated by a vector v^i is called a contra affine motion, a concurrent affine motion, a special concircular affine motion, a recurrent affine motion, a concircular affine motion, a torse forming affine motion and a birecurrent affine motion according as the vector v^i is a contra, concurrent, special concircular, recurrent, concircular, torse forming and birecurrent respectively.

3. Some Lemmas:

This section is devoted to establish two lemmas which are useful in proving the theorems of the next sections.

Lemma 3.1 *For every vector v^i independent of $\dot{x}^i$, the following conditions are equivalent*

$$(3.1) \quad (a) K_{jkh}^i v^h = 0, \quad (b) K_{hjk}^i v^h = 0.$$

Proof. Let $v^i(x^j)$ be any vector satisfying (3.1a). Transvecting the Bianchi identity²⁰

$$(3.2) \quad K_{jkh}^i + K_{hjk}^i + K_{khj}^i = 0$$

by v^h and using (3.1a), we get

$$(3.3) \quad K_{hjk}^i v^h + K_{khj}^i v^h = 0.$$

Using the skew symmetric property of the tensor K_{khj}^i , we have (3.1b).

Conversely, suppose the vector $v^i(x^j)$ satisfies (3.1b). From Bianchi identity (3.2) and (3.1b), we have

$$(3.4) \quad K_{jkh}^i v^h = K_{kjh}^i v^h.$$

Transvecting (3.4) by g_{im} and using (2.11), we get

$$(3.5) \quad K_{jmkh} v^h = K_{kmjh} v^h$$

which, in view of skew symmetry of K_{kmjh} in last two indices, gives

$$(3.6) \quad K_{jmhk} v^h = K_{kmhj} v^h.$$

Thus, we find the tensor $K_{jmhk} v^h$ is symmetric in first and fourth indices.

Transvecting (3.1b) by g_{im} and using (2.11), we get

$$(3.7) \quad K_{hmjk} v^h = 0.$$

We know that the tensor K_{ijkh} satisfies the identity²⁰

$$(3.8) \quad K_{ijhk} - K_{hkij} = \{ C_{ihl} K_{rjk}^l - C_{ikl} K_{rjh}^l + C_{jkl} K_{rih}^l - \\ - C_{jhl} K_{rik}^l + C_{hkl} K_{rij}^l - C_{ijl} K_{rhk}^l \} \dot{x}^r.$$

Transvecting (3.8) by v^h and using (3.7), we have

$$(3.9) \quad K_{ijhk} v^h = v^h \{ (C_{ihl} K_{rjk}^l + C_{jhl} K_{rki}^l + C_{hkl} K_{rij}^l) \\ - (C_{ikl} K_{rjh}^l + C_{jkl} K_{rhi}^l + C_{ijl} K_{rhk}^l) \} \dot{x}^r.$$

Interchanging the indices i and k in (3.9), we have

$$(3.10) \quad K_{kjhi} v^h = v^h \{ (C_{khl} K_{rji}^l + C_{jhl} K_{rik}^l + C_{hil} K_{rkj}^l) - (C_{kil} K_{rjh}^l + C_{jil} K_{rhk}^l + C_{kjl} K_{rhi}^l) \} \dot{x}^r.$$

Adding the corresponding sides of (3.9) and (3.10) and using the symmetry property of the tensor $K_{ijhk} v^h$ in first and fourth indices, we find

$$(3.11) \quad 2K_{ijhk} v^h = -v^h \dot{x}^r \{ 2C_{ikl} K_{rjh}^l + 2C_{jkl} K_{rih}^l + 2C_{jil} K_{rkh}^l \}.$$

Transvecting (3.11) by $\dot{x}^i$ and using (2.9), we get

$$(3.12) \quad 2K_{ijhk} \dot{x}^i v^h = -2C_{jkl} H_{ih}^l v^h \dot{x}^i.$$

Transvection of (3.12) by g^{jp} gives

$$(3.13) \quad K_{ihk}^p \dot{x}^i v^h = -C_{kl}^p H_{ih}^l v^h \dot{x}^i,$$

which, in view of (2.9), yields

$$(3.14) \quad H_{hk}^p v^h = -C_{kl}^p H_{ih}^l v^h \dot{x}^i.$$

Transvecting (3.14) by $\dot{x}^k$ and using (2.4b), we have

$$(3.15) \quad H_{kh}^p v^h \dot{x}^k = 0.$$

From (3.14) and (3.15), we have

$$(3.16) \quad H_{kh}^p v^h = 0.$$

In view of (2.9) and (3.16), (3.11) may be written as

$$(3.17) \quad K_{ijhk} v^h = 0.$$

Transvecting (3.17) by g^{jp} , we get

$$K_{ihk}^p v^h = 0,$$

which, in view of skew symmetry of tensor K_{ihk}^p in its last two indices, gives (3.1a).

Thus, the conditions (3.1a) and (3.1b) are equivalent.

Lemma 3.2 *An infinitesimal transformation is a motion if and only if*

$$(3.18) \quad \mathcal{L} F^2 = 0 .$$

Proof. We know that an infinitesimal transformation is a motion if and only if

$$(3.19) \quad \mathcal{L} g_{ij} = 0 .$$

Thus, to establish the above Lemma, we have to show the equivalence of the conditions $\mathcal{L} F^2 = 0$ and $\mathcal{L} g_{ij} = 0$.

Suppose, we have $\mathcal{L} F^2 = 0$. Differentiating $\mathcal{L} F^2 = 0$ partially with respect to $\dot{x}^i$ and $\dot{x}^j$ and using the commutativity of the operators $\mathcal{L}$ and $\dot{\partial}_i$, we have $\mathcal{L} \dot{\partial}_i \dot{\partial}_j F^2 = 0$; which due to (2.1a), gives $\mathcal{L} g_{ij} = 0$.

Conversely, transvecting $\mathcal{L} g_{ij} = 0$ by $\dot{x}^i$ and $\dot{x}^j$ and using (2.1b), we get $\mathcal{L} F^2 = 0$. This completes the proof.

4. Contra affine motion

Let us consider an infinitesimal transformation generated by a contra vector $v^j(x^j)$ characterized by (2.16a). Differentiating (2.16a) covariantly with respect to x^j , we get

$$(4.1) \quad v^i_{|kj} = 0 .$$

Taking the skew symmetric part of (4.1) and using (2.8), we have (3.1b); which, by Lemma (3.1), implies (3.1a). Writing the commutation formula given by (2.7) for $v^j(x^j)$ and using (2.16a), we have

$$(4.2) \quad (\dot{\partial}_k^* \Gamma_{rh}^j) v^r = 0 .$$

Using (3.1a), (4.1) and (4.2) in (2.14), we have (2.15). Hence the infinitesimal transformation considered is an affine motion. Thus we obtain:

Theorem 4.1 *Every contra vector generates an affine motion in a Finsler space.*

Remark. Theorem (4.1) may also be proved by using Lemma (3.2). In fact the Lie derivative of F^2 is given by $\mathcal{L}F^2 = (\partial_r F^2) v^r_{|s} \dot{x}^s$, which vanishes identically if $v^i_{|k} = 0$ i.e. the vector v^i is a contra vector. Thus, in view of Lemma (3.2) the transformation considered is a motion. Since every motion is an affine motion, the transformation generated by a contra vector is necessarily an affine motion.

Since every affine motion is a projective motion and by Theorem (4.1), every contra vector generates an affine motion, we come to the following conclusion:

Corollary 4.1 *Every contra vector generates a projective motion.*

Since the Lie derivative of the curvature tensor vanishes for an affine motion, in this case we have

$$(4.3) \quad \mathcal{L}K^i_{jkh} = 0.$$

Corollary 4.2 *If any contra vector v^i generates an infinitesimal transformation in a recurrent Finsler space, it must be orthogonal to the recurrence vector.*

Proof. Let us consider a recurrent Finsler space characterized by¹⁶.

$$(4.4) \quad K^i_{jkh|m} = \lambda_m K^i_{jkh}, \quad K^i_{jkh} \neq 0, \quad \lambda_m \neq 0.$$

The non zero vector λ_m appearing in (4.4) is called the recurrence vector. Suppose there is a contra vector v^i (x^j) generating an infinitesimal transformation in the above space. Then the curvature tensor satisfies (4.3). Expanding the left hand side of (4.3) with the help of (2.13) and using (2.16a) (the condition characterizing a contra vector) and (4.4) we get $\lambda_r v^r K^i_{jkh} = 0$. Since $K^i_{jkh} \neq 0$, $v^r \lambda_r = 0$. Thus, we see that the contra vector v^i is orthogonal to the recurrence vector λ_m .

Corollary 4.3 *If a projective recurrent Finsler space admits an infinitesimal transformation generated by a contra vector v^i (x^j), then the vector v^i is orthogonal to its recurrence vector.*

Proof. Let a projective recurrent Finsler space characterized by

$$(4.5) \quad W^i_{jkh|m} = \lambda_m W^i_{jkh}, \quad W^i_{jkh} \neq 0, \quad \lambda_m \neq 0$$

where W_{jkh}^i denotes projective curvature tensor, admits an infinitesimal transformation generated by a contra vector $v^i(x^j)$. By Corollary (4.1), this transformation is a projective motion, and hence the Lie derivative of projective curvature tensor vanishes identically i.e. $\mathcal{L} W_{jkh}^i = 0$. Expanding the left hand side of this equation and using (2.16a) and (4.5), we get $v^m \lambda_m W_{jkh}^i = 0$, which implies $v^m \lambda_m = 0$.

Corollary 4.4 *If a birecurrent finsler space characterized by¹⁷*

$$(4.6) \quad K_{jkhlm}^i = a_{lm} K_{jkh}^i, \quad a_{lm} \neq 0, \quad K_{jkh}^i \neq 0$$

admits an infinitesimal transformation generated by a contra vector $v^i(x^j)$, then the recurrence tensor a_{lm} satisfies

$$(a) \ a_{lm} v^m = 0, \ (b) \ a_{ml} v^m = 0.$$

Proof. If a birecurrent Finsler space characterized by (4.6) admits an infinitesimal transformation generated by contra vector v^i , its curvature tensor satisfies (4.3). Expanding (4.3) with the help of (2.13) and (2.16b), we have

$$(4.8) \quad v^r K_{jkhlr} = 0.$$

Differentiating (4.8) covariantly with respect to x^l and using (2.16a), we get

$$v^r K_{jkhrl} = 0,$$

which, in view of (4.6), implies (4.7a). Taking skew symmetric part of (4.6) with respect to indices l and m and using (2.8), we get

$$(4.9) \quad K_{jkh}^r K_{rml}^i - K_{rkh}^i K_{jml}^r - K_{jrh}^i K_{kml}^r - K_{jkr}^i K_{hml}^r - (\partial_r K_{jkh}^i) K_{sml}^r \dot{x}^s \\ = (a_{lm} - a_{ml}) K_{jkh}^i.$$

While proving Theorem (5.1), we have seen that (3.1a) and (3.1b) hold. Transvecting (4.9) by v^m and usig (3.1a), (3.1b) and (4.7a), we have (4.7b).

5. Concurrent affine motion

Let us consider a Finsler space admitting an infinitesimal transformation generated by concurrent vector $v^i(x^j)$ characterized by (2.16b). Differentiating (2.16b) covariantly with respect to x^j , we have (4.1). Taking the skew symmetric part of (4.1) and using (2.8), we get (3.1b); which by Lemma 3.1, implies (3.1a). It follows from (2.14), (4.2), (3.1a) and (4.1) that the Lie derivative of Γ_{jk}^{*i} vanishes identically. Hence the infinitesimal transformation considered is an affine motion. Thus, we have

Theorem 5.1 *If a Finsler space admits an infinitesimal transformation generated by a concurrent vector, the transformation is necessarily an affine motion.*

However, a concurrent vector does not generate a motion in a Finsler space because in this case $\mathcal{L}F^2 = 2CF^2$, which cannot vanish unless $C = 0$.

Since the Lie derivative of the curvature tensor vanishes with respect to an affine motion, we have (4.3). Expanding (4.3) with the help of (2.13) and (2.16b), we get

$$(5.1) \quad v^r K_{jkhlr}^i + 2CK_{jkh}^i = 0;$$

which, after covariant differentiation, gives

$$(5.2) \quad v^r K_{jkhlrl}^i + 3CK_{jkhl}^i = 0.$$

Differentiating (3.1a) covariantly, we get

$$(5.3) \quad v^h K_{jkhlm}^i + CK_{jkm}^i = 0.$$

Thus, we obtain:

Theorem 5.2 *If a Finsler space admits an infinitesimal transformation generated by a concurrent vector characterized by (2.13b), its curvature tensor satisfies (5.1)–(5.3).*

Let the space considered be a recurrent space characterized by (4.4). By Theorem 5.2, it admits (5.3). In view of (3.1a) and (4.4), equation (5.3) gives $CK_{jkh}^i = 0$; which implies either $C = 0$ or $K_{jkh}^i = 0$. Both the conditions contradict our hypothesis. Hence we conclude:

Corollary 5.1 *A recurrent Finsler space does not admit any infinitesimal*

transformation generated by a concurrent vector.

If the space considered is a symmetric space characterized by

$$(5.4) \quad K_{jkhlm}^i = 0,$$

the equation (5.3) gives $CK_{jkh}^i = 0$; which implies $K_{jkh}^i = 0$ for $C \neq 0$. Thus we see that a symmetric space admitting an infinitesimal transformation generated by a concurrent vector is necessarily flat. This leads to the following:

Corollary 5.2 *A non flat symmetric Finsler space does not admit any infinitesimal transformation generated by a concurrent vector.*

If the space considered is a birecurrent space characterized by (4.6), it admits (5.3). Differentiating (5.3) covariantly with respect to x^l , we get

$$(5.5) \quad v^h K_{jkhlm}^i + CK_{jklm}^i + CK_{jkm}^i = 0.$$

Using (3.1a) and (4.7) in (5.5), we get

$$(5.6) \quad K_{jklm}^i + K_{jkm}^i = 0.$$

Transvecting (5.6) by v^l using (5.1) and (5.3), we have $CK_{jkm}^i = 0$, which implies $K_{jkh}^i = 0$, a contradiction. This leads to:

Corollary 5.3 *A birecurrent Finsler space does not admit any infinitesimal transformation generated by a concurrent vector.*

Let us consider a bisymmetric Finsler space characterized by¹³

$$(5.7) \quad K_{jkhlm}^i = 0.$$

If this space admits an infinitesimal transformation generated by a concurrent vector $v^i(x^j)$, then its curvature tensor satisfies (5.2) and (5.3). In view of (5.7), equation (5.2) reduces to $CK_{jkhll}^i = 0$, which implies $K_{jkhll}^i = 0$ as $C \neq 0$. Using $K_{jkhll}^i = 0$ in (5.3), we get $K_{jkh}^i = 0$. Hence a bisymmetric Finsler space admitting an infinitesimal transformation generated by a concurrent vector is flat. Thus, we have:

Corollary 5.4 *A non flat bisymmetric Finsler space does not admit any*

transformation generated by a concurrent vector.

6. Special Concircular affine motion

Let us consider a Finsler space admitting an infinitesimal transformation generated by a special concircular vector characterized by (2.16c). If this transformation is an affine motion, we have $\mathcal{L}\Gamma_{jk}^* = 0$, which by (2.14) and (2.16c) gives

$$(6.1) \quad \rho_{lk} \delta_j^i + K_{jkh}^i v^h = 0.$$

Transvecting (6.1) by $\dot{x}^j$ and using (2.9), we have

$$(6.2) \quad \rho_{lk} \dot{x}^i + H_{kh}^i v^h = 0.$$

Transvecting (6.2) by y_i and (2.2b) and (2.10), we have $F^2 \rho_{lk} = 0$, which implies $\rho_{lk} = 0$. Thus we get a contradiction. Therefore we may conclude:

Theorem 6.1 *An infinitesimal transformation generated by a special concircular vector cannot be an affine motion in a Finsler space. In other words, a Finsler space does not admit any special concircular affine motion.*

7. Recurrent affine motion

Let us consider an infinitesimal transformation generated by a recurrent vector characterized by (2.16d). In view of (2.14) and (2.16d), we may write the expression for the Lie derivative of Γ_{jk}^* as follows:

$$(7.1) \quad \mathcal{L}\Gamma_{jk}^* = (\mu_{jk} + \mu_j \mu_k) v^i + K_{jkh}^i v^h + (\partial_h \Gamma_{jk}^*) \mu v^h,$$

where $\mu = \mu_s \dot{x}^s$. Writing the commutation formula given by (2.7) for the vector v^i , and using (2.16d) and the fact that v^i is independent of $\dot{x}^i$, we may easily have

$$(7.2) \quad (\partial_j \Gamma_{rk}^*) v^r = \partial_j \mu_k v^j.$$

Transvecting (7.2) by $\dot{x}^k$ and using the equation,²⁰

$$(7.3) \quad \dot{x}^k \dot{\partial}_j \Gamma_{rk}^* = C_{rjll}^i \dot{x}^l,$$

we get

$$(7.4) \quad C_{rjll}^i \dot{x}^i v^r = (\dot{x}^k \dot{\partial}_j \mu_k) v^i.$$

Transvecting (7.4) by y_i and using $y_i C_{rj}^i = 0$ and $y_{ill} = 0$, we have $(y_i v^i) \dot{x}^k \dot{\partial}_j \mu_k = 0$. Since $y_i v^i = 0$ implies $v^i = 0$ ¹⁶, the condition $(y_i v^i) \dot{x}^k \dot{\partial}_j \mu_k = 0$ implies

$$(7.5) \quad \dot{x}^k \dot{\partial}_j \mu_k = 0$$

which implies

$$(7.6) \quad \mu_j = \dot{\partial}_j \mu,$$

Now from equation (7.3), we get

$$\dot{x}^j \dot{\partial}_r \Gamma_{ij}^* = C_{irlj}^h \dot{x}^j = (\dot{\partial}_i \Gamma_{rj}^*) \dot{x}^j.$$

Transvecting this equation by v^r

$$v^r (\dot{\partial}_r \Gamma_{ij}^*) \dot{x}^j = (C_{irlj}^h) v^r \dot{x}^j = (\dot{\partial}_i \Gamma_{rj}^*) \dot{x}^j v^r.$$

Which, in view of (7.2) and (7.5), gives

$$(7.7) \quad v^r (\dot{\partial}_r \Gamma_{ij}^*) \dot{x}^j = 0.$$

If the transformation considered is an affine motion, the Lie derivative of Γ_{jk}^* vanishes identically. Therefore the equation (7.1) reduces to

$$(7.8) \quad (\mu_{jlk} + \mu_j \mu_k) v^i + K_{jkh}^i v^h + \mu v^h (\dot{\partial}_h \Gamma_{jk}^*) = 0.$$

Now transvecting (7.8) by $\dot{x}^j$ and using (2.9) and (7.7), we have

$$(7.9) \quad (\mu_{lk} + \mu \mu_k) v^i + H_{kh}^i v^h = 0.$$

Transvecting equation (7.9) by y_i and using (2.10), we get

$$(7.10) \quad \mu_{lk} + \mu \mu_k = 0,$$

since $y_i v^i \neq 0$. Using (7.10) in (7.9), we get $H_{kh}^i v^h = 0$. Differentiating (7.10) with respect to x^j and using commutation formula, we get

$$(7.11) \quad \mu_{jlk} - \mu_r (\dot{\partial}_j \Gamma_{mk}^r) \dot{x}^m + \mu_j \mu_k + \mu \dot{\partial}_j \mu_k = 0.$$

Taking skew symmetric part of (7.11) and using the symmetric property of $(\dot{\partial}_j \Gamma_{mk}^r) \dot{x}^m$ and $\dot{\partial}_j \mu_k$ in the indices j and k , we have

$$(7.12) \quad \mu_{jlk} = \mu_{klj}.$$

Differentiating (2.16d) covariantly with respect to x^j and taking its skew symmetric part, we have

$$v_{lkj}^i - v_{lyk}^i = 0;$$

which, in view of the commutation formula (2.8), gives $K_{hjk}^i v^h = 0$. In view of the Lemma 3.1 the equation $K_{hjk}^i v^h = 0$ implies $K_{jkh}^i v^h = 0$. Using (7.11) and $K_{jkh}^i v^h = 0$ in (7.8), we get

$$v^i \mu_r (\dot{\partial}_j \Gamma_{mk}^r) \dot{x}^m - \mu (\dot{\partial}_j \mu_k) v^i + \mu v^h (\dot{\partial}_h \Gamma_{jk}^i) = 0;$$

which in view of (7.2), implies

$$(7.13) \quad v^i \mu_r (\dot{\partial}_j \Gamma_{mk}^r) \dot{x}^m + \mu v^h (\dot{\partial}_h \Gamma_{jk}^i - \dot{\partial}_j \Gamma_{hk}^i) = 0.$$

Multiplying this equation by y_i and using $y_i (\dot{\partial}_h \Gamma_{jk}^i - \dot{\partial}_j \Gamma_{hk}^i) = 0$, we get

$(y_i v^i) \mu_r (\dot{\partial}_j \Gamma_{mk}^r) \dot{x}^m = 0$, which implies $\mu_r (\dot{\partial}_j \Gamma_{mk}^r) \dot{x}^m = 0$. Therefore the equation (7.11) and (7.13) reduce to

$$(7.14) \quad \mu_{jk} + \mu_j \mu_k + \mu \dot{\partial}_j \mu_k = 0.$$

and

$$(7.15) \quad v^h (\dot{\partial}_h \Gamma_{jk}^i - \dot{\partial}_j \Gamma_{hk}^i) = 0.$$

since $\mu \neq 0$. Thus, we see that if a recurrent vector v^i characterized by (2.16d) generates an affine motion in a Finsler space, the conditions (7.14) and (7.15) necessarily hold. Conversely, suppose the conditions (7.14) and (7.15) hold for a recurrent vector v^i characterized by (2.16d). Differentiating (2.16d) covariantly, taking its skew symmetric part and using the commutation formula (2.8) and the equation (7.14), we get $K_{hjk}^i v^h = 0$. Using Lemma 3.1, we conclude $K_{jkh}^i v^h = 0$. Using this

equation, the equations (7.14), (7.15) and (7.2), we have $\mathcal{L} \Gamma_{jk}^i = 0$. Thus, the transformation considered is an affine motion. Therefore we may conclude:

Theorem 7.1 *An infinitesimal transformation generated by a recurrent vector defined as (2.16d) is an affine motion if and only if the conditions (7.14) and (7.15) hold good.*

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