

A Study of Hairy Ball Theorem associate with Borsuk-Ulam Theorem

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Abstract: In this paper we will provide shorter proof of Hairy Ball theorem using a geometric approach and orientation of curve. Using idea from Borsak-Ulam theorem we will provide few many lemmas and application of Hairy ball theorem.

Keywords: Borsum-Ulam Theorem, vector bundle, Winding number and Hairy Ball Theorem.

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1. Introduction

In Borsuk-Ulam theorem for any continuous function $f: S^n \rightarrow \mathbb{R}^n$ there exists a $x \in \mathbb{R}^n$ such that $f(-x) = f(x)$. In Hairy ball theorem if $f: S^n \rightarrow \mathbb{R}^3$ is a continuous vector field then there exists $x \in S^n$ such that $f(x) = 0$, when n is even. It is interesting to study Hairy ball theorem in view of Borsuk-Ulam theorem and see geometrically what obstacle is coming to Hairy Ball theorem for which restriction is given on n in Hairy Ball theorem.

Now we recall the following definition and statements:

Definition 1.1: A function $f: S^2 \rightarrow \mathbb{R}^3$ is said to be a vector field on S^2 if $\langle v(x), x \rangle = 0$ for all $x \in S^2$ and the vector field is continuous if its component function is continuous.

Definition 1.2: A regular smooth curve on a plane is a smooth map whose derivative does not vanishes anywhere.

Definition 1.3: Let $\gamma: [0,1] \rightarrow \mathbb{C} \setminus \{0\}$ be a continuous closed path on the plane minus one point. The Winding number of γ around a is the integer, $\text{wind}(\gamma, a) = s(1) - s(0)$

Definition 1.4: (de Rham cohomology) For the S^n sphere, and also when taken together with a product of open intervals, we have the following,

Let $n > 0, m \geq 0$ and I be an open real interval. Then

$$H_{dR}^k(S^n \times I^m) \simeq \begin{cases} \mathbb{R}, & \text{for } k = 0 \text{ or } k = n \\ 0, & \text{if } k \neq 0 \text{ and } k \neq n \end{cases}$$

Definition 1.5 : (Borsuk-Ulam Theorem) If $f: S^n \rightarrow \mathbb{R}^n$ is continuous map then there exists an $x \in S^n$ such that $f(-x) = f(x)$.

Theorem 1.1 : (Stoke's theorem) Suppose S is a surface in $\mathbb{R}^3$ which is positively oriented and its boundary is given by $\delta S = \eta$. Then for a vector field $F(x, y, z) = (F_x(x, y, z), F_y(x, y, z), F_z(x, y, z))$ defined over the region S and have all three continuous first order partial derivatives in a region containing S ,

$$\iint (\nabla \times F) \cdot dS = \oint F \cdot d\eta$$

2. Main Results

Theorem 2.1: A smooth vector field on S^n must vanish somewhere if n is even.

Proof : Let n is even and $\vec{v}$ is nowhere vanishing smooth vector field on S^n . Corresponding to each $p \in S^n$ define $\gamma_p: [0, \frac{\pi}{||\vec{v}(p)||}] \rightarrow S^n$ be the smooth parametric great circle (with constant speed) going from p to $-p$ and the velocity vector $\gamma'_p(0) = \vec{v}(p) \neq 0$. The formula for the parametric curve $\gamma_p(t)$ is,

$$\gamma_p(t) = \cos(t||\vec{v}(p)||)p + \sin(t||\vec{v}(p)||)\frac{\vec{v}(p)}{||\vec{v}(p)||} \in S^n \subseteq \mathbb{R}^{n+1}$$

Consider the mapping $F: S^n \times [0, 1] \rightarrow S^n$ defined by,

$$F(p, t) = \gamma_p\left(\frac{\pi t}{||\vec{v}(p)||}\right)$$

The map F is smooth and satisfies $F(p, 0) = p$ and $F(p, 1) = -p$ for all $p \in S^n$. Hence F defines a smooth homotopy from the identity map on S^n to the antipodal map $p \rightarrow -p$ on S^n (and it is continuous homotopy if $\vec{v}$ is merely continuous and nowhere vanishing).

Since so far the fact that n is even is not used so to prove the hairy ball theorem, it is enough to prove that if n is even then the identity map antipodal map $S^n \rightrightarrows S^n$ are not smoothly homotopic to each other. By homotopy-invariance, we get that under the existence of such a $\vec{v}$ the antipodal map $A: S^n \rightarrow S^n$ induces the identity map $A^*: H_{dR}^k(S^n) \rightarrow H_{dR}^k(S^n)$ on the k -th deRham cohomology of S^n for all $k \geq 0$. For $k = n$ to get a contradiction it

is enough to prove that if n is even then A^* as a self-map of $H_{dR}^k(S^n)$ is not the identity map.

Consider the n -form on $\mathbb{R}^{n+1}$ defined by,

$$\omega = \sum_{i=1}^n (-1)^{i-1} x_i dx_1 \wedge \dots \wedge dx_i \wedge \dots \wedge dx_{n+1}$$

Clearly $d\omega = (n+1)dx_1 \wedge \dots \wedge dx_{n+1}$ and so for the unit ball $B^{n+1} \subseteq \mathbb{R}^{n+1}$ with its standard orientation we have,

$$\int_{B^{n+1}} d\omega = (n+1)Vol(B^{n+1}) \neq 0$$

By Stokes theorem for B^{n+1} , if $\eta = \omega|_{S^n}$ and we give $S^n = \partial B^{n+1}$ the induced boundary orientation then $\int_{S^n} \eta = \int_{B^{n+1}} d\omega \neq 0$. Hence by stokes theorem for the boundary less smooth compact oriented manifold S^n . We conclude that the top-degree differential form η on S^n is not exact. That is, its deRham cohomology class $[\eta] \in H_{dR}^n(S^n)$ is non-zero.

By the existence of the smooth homotopy between A and the identity map, it follows that A^* on $H_{dR}^n(S^n)$ is the identity map, so $[A^*(\eta)] = A^*([\eta])$ is equal to $[\eta]$. That is the top degree differential form $A^*(\eta)$ and η differ by an exact form. But the antipodal map $A: S^n \rightarrow S^n$ is induced by the negation map $: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$, and by inspection of the definition of $\omega \in \Omega_{\mathbb{R}^{n+1}}^n$ we have

$$N^*(\omega) = (-1)^{n+1}$$

Hence, pulling back this equality to the sphere gives $A^*(\eta) = (-1)^{n+1} \cdot \eta$ in $\Omega_{S^n}^n(S^n)$. Thus, in $H_{dR}^n(S^n)$ we have

$$[\eta] = A^*([\eta]) = [A^*(\eta)] = [(-1)^{n+1}\eta] = (-1)^{n+1}[\eta]$$

If n is even we therefore have $[\eta] = -[\eta]$, so $[\eta] = 0$. But we have already seen via Stokes' theorem for the boundary less manifold S^n and for the manifold with boundary B^{n+1} that $[\eta]$ is the non-zero. This completes the proof.

Theorem 1.12 : *There is no non-vanishing continuous tangent vector field on 2-dimensional.*

Proof: Let $\vec{v}$ be a non-vanishing continuous vector field on S^2 that is $\|\vec{v}\| \neq 0$ and so $\frac{1}{\|\vec{v}\|} \vec{v}$ makes sense and $\frac{1}{\|\vec{v}\|} \vec{v}$ is a vector of length 1.

Consider the following orientation on $\mathbb{R}^3$:

Let $\{e_1, e_2, e_3\}$ is a basis of $\mathbb{R}^3$ which is positively oriented. Now consider $\mathbb{R}^2$ as a subset of $\mathbb{R}^3$ in following way $\mathbb{R}^2 = \{(x, y, 0) : x, y \in \mathbb{R}\} \subseteq \mathbb{R}^3$. A orientation and rotation number of a curve η on S^2 can be provide as,

$$n(\eta) = \frac{1}{2\pi} \times \text{change of oriented angle respect a variable direction.}$$

Corresponding to each $x \in S^2$ and for each $\frac{1}{\|w\|}w$ from the tangent vector field of x there exist unique $\frac{1}{\|w^\perp\|}w^\perp$ from the tangent vector field of x so that $\{x, \frac{1}{\|w\|}w, \frac{1}{\|w^\perp\|}w^\perp\}$ have positively orientation (respect to the orientation we defined).

For each such pair $\{x, w\}$ an isometry on $\mathbb{R}^3$ can be defined in following way:

$$p \rightarrow \left\{ x, \frac{1}{\|w\|}w, \frac{1}{\|w^\perp\|}w^\perp \right\} \rightarrow \{e_1, e_2, e_3\}$$

As p, w changes continuously this map will be a continuous map. For each $x \in S^2$ and $t \in (-1, 1)$ defined the smooth curve on S^2 as,

$$\gamma_{x,t} = S^2 \cap \{y \in S^2: \langle x, y \rangle = t\}$$

Precisely this is a circle on S^2 whose center is lies on the line joining x and origin . Also, the center is at a distance $|t|$ from origin and $1 - |t|$ from x .

In particular for the choice of $x = (1, 0, 0), t = \frac{1}{2}, t = 1/2$ and $x = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ the picture is as follows :

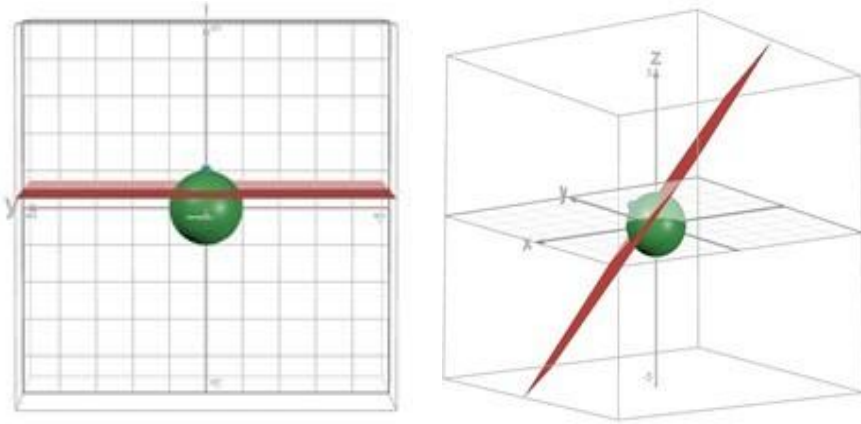


Figure 1: Circle S^2 with radius $1/2$

In particular, if we choose $t = 0$ then the plane $\{y \in S^2: \langle x, y \rangle = t\}$ passes through origin and so the smooth curve $\gamma_{x,t}$ becomes the greatest circle on S^2 . For two points x and $-x$ on S^2 the smooth curve $\gamma_{x,t}$ and $\gamma_{-x,t}$ are same great circle but different orientation. This gives,

$$\begin{aligned} n(\gamma) &= -n(\gamma) \\ \Rightarrow n(\gamma) &= 0 \end{aligned}$$

If we choose $t \rightarrow 1$ then the value $n(\gamma)$ tends to rotation number of the circle in the plane and so $n \in \{-1, 1\}$ This leads us to a contradiction.

Obstacle when n is odd : If n is odd we define a map $f: S^n \rightarrow \mathbb{R}^{n+1}$ as $f(x_1, x_2, \dots, x_n) = (x_2, -x_1, \dots, x_n, -x_{n-1})$. Because of the dot product $(x_1, x_2, \dots, x_n) \cdot (x_2, -x_1, \dots, x_n, -x_{n-1}) = 0$, this is a specific example of vanishing vector field on S^n .

This is why Hairy ball theorem is specifically for even dimension. When n is odd there are even number of variable and we can have $\frac{n+1}{2}$ number of pair and setting up each pair in such a way their dot product is zero the vector field vanishes at some point. But when n is even we cannot partition variables by taking two variable in each partition. This is how the Hairy ball theorem depends on n .

Comparison : A consequence of Hairy Ball theorem is for any continuous function $f: S^{2n} \rightarrow S^{2n}$ satisfies either $f(x) = x$ or $f(x) = -x$ for some $x \in S^{2n}$. So if the function does not have a fix point there exist a point on the sphere that maps to a antipodal point.

Using Homotopy :

Suppose $f: S^2 \rightarrow \mathbb{R}^3$ be a continuous vector field and v_x is the vector at the point x . If we define a homotopy $H: S^2 \times [0,1] \rightarrow \mathbb{R}^3$ by $H(s, t)$ is the point $t\pi$ radians away from x along the great circle defined by v_x then this will be a homotopy between identity and antipodal map on S^2 .

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References

1. M. Eisenberg, R. Guy; A proof of the Hairy Ball theorem, *Amer. Math. Monthly*, 86 (1979), 571-574.
2. M. R. Adhikari; *Basic Algebraic Topology and its Applications*, Springer, 2016.
3. E. H. Spanier; *Algebraic Topology*, Mc Graw-Hill, New York.
4. Algebraic Topology: Beckham Mayers lecture notes.
5. John M. Lee; *Introduction to Smooth Manifolds*, 2003.