

Para Sasakian Manifold Admitting Ricci Solitons

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Abstract: In this paper, our aim is to investigate para-Sasakian manifold admitting Ricci solitons satisfying the conditions $R(\xi, X).R = 0$, $R(\xi, X).P = 0$, $P(\xi, X).R = 0$ and $S(\xi, X).P = 0$ and also find some interesting results.

Keywords: Para-Sasakian manifolds, Ricci solitons.

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1. Introduction

T. Takahashi¹ introduced almost contact manifolds equipped with an associated pseudo-Riemannian metric. In particular, he studied Sasakian manifolds equipped with an associated pseudo-Riemannian metric. T. Adati and K. Matsumoto² defined para-Sasakian and special para-Sasakian manifolds. These manifolds are special cases of an almost paracontact manifold introduced by I. Sato and K. Matsumoto³. In the same paper, the authors studied conformally symmetric para-Sasakian manifolds and they proved that an n -dimensional conformally symmetric para-Sasakian manifold is conformally flat and special para-Sasakian. Later on, Ricci solitons in contact metric manifolds have been studied by several authors like Bagewadi et al.⁴⁻⁶, Bejan and Crasmareanu⁷, Blaga⁸⁻¹⁰, Chandra et al.¹¹, Chen and Deshmukh¹², Chenxu and Meng¹³, Nagaraja and Premalatha¹⁴. η -Ricci soliton have been studied by various authors¹⁵⁻¹⁹.

The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}$$

A Ricci soliton (g, V, λ) on a Riemannian manifold (M, g) is generalization of an Einstein metric such that it satisfies the following condition²⁰

$$(1.1) \quad L_V g(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0,$$

where S is the Ricci tensor, L_V is the Lie derivative operator along the vector field V on (M, g) and λ is a real number. The Ricci soliton is said to be shrinking, steady, or expanding according as λ is negative, zero or positive.

The present work is organized as follows: After introduction, in section 2, we recall basic formulas and results of para-Sasakian manifold. Section 3, deals with Ricci soliton in para-Sasakian manifolds satisfying $R(\xi, X).R = 0$. Section 4 concerned with the study of Ricci soliton in para-Sasakian manifolds satisfying $R(\xi, X).P = 0$. In section 5, we study Ricci soliton in para-Sasakian manifolds satisfying $P(\xi, X).R = 0$. Section 6 is devoted to the study of Ricci soliton in para-Sasakian manifolds satisfying $S(\xi, X).P = 0$.

2. Preliminaries

An n -dimensional differentiable manifold M is called almost paracontact manifold with almost paracontact structure (φ, ξ, η) consisting of a $(1,1)$ -tensor field φ , a vector field ξ , η is 1-form and semi-Riemannian metric g such that

$$(2.1) \quad \varphi^2(X) = X - \eta(X)\xi,$$

$$(2.2) \quad \eta(\xi) = 1,$$

$$(2.3) \quad \varphi(\xi) = 0,$$

$$(2.4) \quad \eta(\varphi X) = 0,$$

$$(2.5) \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$(2.6) \quad g(\varphi X, Y) = g(X, \varphi Y),$$

$$(2.7) \quad g(X, \xi) = \eta(X),$$

$$(2.8) \quad g(\xi, \xi) = \eta(\xi) = 1,$$

$$(2.9) \quad (\nabla_X \varphi)Y = -g(X, \varphi Y)\xi - \eta(Y)\varphi^2 X,$$

$$(2.10) \quad (\nabla_X \xi) = \varphi(X),$$

where ∇ is the Levi-Civita connection. A manifold endowed with para-Sasakian structure $(M, \varphi, \xi, \eta, g)$ is called a para-Sasakian manifold.

$$(2.11) \quad d\eta(X, Y) = \Omega(X, Y) = (\nabla_X \eta)Y = g(\varphi X, Y),$$

for all X, Y, Z in M , where Ω is the fundamental 2-form.

In a para-Sasakian manifold M , the following relations hold

$$(2.12) \quad R(\xi, X)Y = -g(X, Y)\xi + \eta(Y)X,$$

$$(2.13) \quad R(X, Y)\xi = -\eta(Y)X + \eta(X)Y,$$

$$(2.14) \quad \eta(R(X, Y)Z) = [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)],$$

$$(2.15) \quad S(\varphi Y, \varphi Z) = S(Y, Z) + (n-1)\eta(Y)\eta(Z),$$

$$(2.16) \quad S(\varphi Y, Z) = S(Y, \varphi Z),$$

$$(2.17) \quad S(Y, \xi) = -(n-1)\eta(Y).$$

Let (g, V, λ) be a Ricci soliton in a para-Sasakian manifold, then

$$(2.18) \quad (L_\xi g)(X, Y) = 2g(X, \varphi Y),$$

Using (2.18) in (1.1), we have

$$(2.19) \quad S(X, Y) = -[g(X, \varphi Y) + \lambda g(X, Y)],$$

$$(2.20) \quad S(X, \xi) = -\lambda \eta(X),$$

$$(2.21) \quad Q(X) = \varphi X - \lambda X,$$

where Q is the Ricci operator.

$$(2.22) \quad r = n[\varphi X - \lambda X].$$

The Projective curvature tensor P is defined as

$$(2.23) \quad P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}[S(Y, Z)X - S(X, Z)Y].$$

By the virtue of (2.14), (2.20) and (2.22), we have

$$(2.24) \quad P(\xi, Y)Z = -g(Y, Z)\xi + \eta(Z)Y + \frac{\lambda}{n-1}[\eta(Z)\xi - g(Y, Z)\xi] - \frac{1}{n-1}g(\varphi Y, Z)\xi.$$

Replacing Y by ξ in (2.23) and using (2.2), (2.14), (2.20) and (2.21), we have

$$(2.25) \quad P(Y, \xi)Z = \left[1 + \frac{\lambda}{n-1}\right][\eta(Z)X - g(X, Z)\xi] + \frac{1}{n-1}g(\varphi X, Z)\xi.$$

Again replacing Z by ξ in (2.23) and using (2.2), (2.3), (2.14), (2.18) and (2.21), we have

$$(2.26) \quad P(X, Y)\xi = \left[1 - \frac{\lambda}{(n-1)}\right][\eta(X)Y - \eta(Y)X],$$

$$(2.27) \quad \begin{aligned} \eta(P(X, Y)Z) &= \left[\frac{\lambda}{(n-1)} - 1\right][g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] \\ &\quad - \frac{1}{n-1}[g(\varphi X, Z)\eta(X) - g(\varphi X, Z)\eta(Y)] \end{aligned}$$

3. Ricci soliton in para-Sasakian manifolds Satisfying $R(\xi, X).R = 0$

In this section, we study Ricci soliton in para-Sasakian manifolds satisfying $R(\xi, X).R = 0$. i.e. $(R(\xi, X).R)(Y, Z)W = 0$, it implies that

$$(3.1) \quad R(\xi, X)R(Y, Z)U - R(R(\xi, X)Y, Z)U - R(Y, R(\xi, X)Z)U \\ - R(Y, Z)(\xi, X)U = 0.$$

Using (2.14) in (3.1), we have

$$(3.2) \quad \eta(R(Y, Z)U)X - g(X, R(Y, Z)U)\xi - \eta(Y)R(X, Z)U + g(X, Y)R(\xi, Z)U \\ - \eta(Z)R(Y, X)U + g(X, Z)R(Y, \xi)U - \eta(U)R(Y, Z)X + g(X, U)R(Y, Z)\xi = 0.$$

Taking inner product of (3.2) with ξ and using (2.2) and (2.3), we have

$$(3.3) \quad g(X, R(Y, Z)U) = \eta(X)\eta(R(Y, Z)U)X - \eta(Y)\eta(R(X, Z)U) . \\ + g(X, Y)\eta(R(\xi, Z)U) - \eta(Z)\eta(R(Y, X)U) \\ + g(X, Z)\eta(R(Y, \xi)U) - \eta(U)\eta(R(Y, Z)X) + g(X, U)\eta(R(Y, Z)\xi).$$

In view of (2.14), equation (3.3) turns to

$$(3.4) \quad g(X, R(Y, Z)W) = -2g(Z, U)\eta(X)\eta(Y) + g(X, Z)g(Y, U)g(X, Y)g(Z, U).$$

Taking $X = U = e_i$ in above equation and taking summation over i , $i \in [1, n]$, we get

$$(3.5) \quad S(Y, Z) = -2n\eta(Y)\eta(Z).$$

Now taking $Y = \xi$ in (3.5), we have

$$(3.6) \quad S(\xi, Z) = -2n\eta(Z).$$

Using (2.19) in (3.6), we have

$$(3.7) \quad \lambda = 2n.$$

Hence we have the following theorem:

Theorem 3.1. *Ricci soliton in para-Sasakian manifolds satisfying the condition $R(\xi, X).R = 0$ is always expanding.*

4. Ricci soliton in para-Sasakian manifolds Satisfying $R(\xi, X).P = 0$

Let us take $R(\xi, X).P = 0$, then we have

$$(4.1) \quad R(\xi, X).P(Y, Z)U - P(R(\xi, X)Y, Z)U - P(Y, R(\xi, X)Z)U \\ - P(Y, Z)R(\xi, X)U = 0.$$

By virtue of (2.12) in (4.1), we have

$$(4.2) \quad \eta(P(Y, Z)U)X - g(X, P(Y, Z)U)\xi - \eta(Y)P(X, Z)U \\ + g(X, Y)P(\xi, Z)U - \eta(Z)P(Y, X)U + g(X, Z)P(Y, \xi)U \\ - \eta(U)P(Y, Z)X + g(X, U)P(Y, Z)\xi = 0.$$

Taking the inner product with ξ in (4.2) and using (2.7) (2.8) we have

$$(4.3) \quad \eta(X)\eta(P(Y, Z)U) - g(X, P(Y, Z)U) - \eta(Y)\eta(P(X, Z)U) \\ + g(X, Y)\eta(P(\xi, Z)U) - \eta(Z)\eta(P(Y, X)U) + g(X, Z)\eta(P(Y, \xi)U) \\ - \eta(U)\eta(P(Y, Z)X) + g(X, U)\eta(P(Y, Z)\xi) = 0.$$

Using (2.27) in (4.3), we get

$$(4.4) \quad g(X, P(Y, Z)U) = \left[\frac{\lambda}{(n-1)} - 1 \right] [g(X, U)\eta(Y) - g(Y, U)\eta(X)]\eta(Z) \\ + [g(X, Y)g(Z, W) - g(Y, U)g(X, Z)] - \frac{1}{(n-1)} [(g(X, Y)g(\varphi Z, U) \\ - g(X, Z)g(\varphi Y, U)) + (g(X, \varphi Y)\eta(Z) - g(\varphi Z, X)\eta(U)] = 0.$$

In view of (2.23), (4.4) reduces to

$$(4.5) \quad g(X, R(Y, Z)U) = \left[\frac{\lambda}{(n-1)} - 1 \right] [g(X, U)\eta(Y) - g(Y, U)\eta(X)]\eta(Z) \\ + [g(X, Y)g(Z, W) - g(Y, U)g(X, Z)] - \frac{1}{(n-1)} [(g(X, U)\eta(Y)\eta(\varphi Z) \\ - g(X, U)\eta(\varphi Y)\eta(Z))] + [g(X, Y)g(\varphi Z, U) - g(X, Z)g(\varphi Y, U) \\ + g(X, \varphi Y)\eta(Z) - g(\varphi Z, X)\eta(U)] \\ + \frac{1}{(n-1)} [S(Z, U)g(X, Y) - S(Y, U)g(X, Z)] = 0.$$

Taking $X = U = \xi$ in (4.5) and using (2.4), (2.7), (2.8) and (2.17), we get

$$(4.6) \quad S(Y, Z) = 0.$$

By using (2.19), (4.6) becomes

$$(4.7) \quad -g(\varphi Y, Z) + \lambda g(Y, Z) = 0.$$

Taking $Y = Z = \xi$ in (4.7) and using (2.3) and (2.8), we get

$$(4.8) \quad \lambda = 0.$$

Hence we can leads to the following theorem:

Theorem 4.1. Ricci soliton in para-Sasakian manifolds satisfying the condition $R(\xi, X).P = 0$ is always steady.

5. Ricci soliton in para-Sasakian manifolds Satisfying $P(\xi, X).R = 0$

In this section, we take $P(\xi, X).R = 0$, then we have

$$(5.1) \quad P(\xi, X)R(Y, Z)W - R(P(\xi, X)Y, Z)W - R(Y, P(\xi, X)Z)W \\ - P(Y, Z)R(\xi, X)W = 0.$$

By virtue of (2.24), equation (5.1) becomes

$$(5.2) \quad [\eta(R(Y, Z)U)X - g(X, R(Y, Z)U)\xi + g(X, Y)R(\xi, Z)U \\ - \eta(Y)R(X, Z)U + g(X, Z)R(Y, \xi)U - \eta(Z)R(Y, X)U] \\ + \frac{\lambda}{(n-1)} [\eta(R(Y, Z)U)\xi - g(X, R(Y, Z)U)\xi - \eta(Y)R(\xi, Z)U \\ + g(X, Y)R(\xi, Z)U + g(X, Z)R(Y, \xi)U - \eta(Z)R(Y, X)U \\ + g(X, Y)R(Y, Z)\xi - \eta(U)R(Y, Z)\xi] - \frac{1}{(n-1)} [g(\varphi X, R(Y, Z)U)\xi \\ - g(\varphi X, Y)R(\xi, Z)U - g(\varphi X, Z)R(Y, \xi)U - g(\varphi X, U)R(Y, Z)\xi] = 0.$$

Taking the inner product with ξ in (5.2) and using (2.7) (2.8) we have

$$(5.3) \quad [\eta(X)\eta(R(Y, Z)U) - g(X, R(Y, Z)U) + g(X, Y)\eta(R(\xi, Z)U) \\ - \eta(Y)\eta(R(X, Z)U) + g(X, Z)\eta(R(Y, \xi)U) - \eta(Z)\eta(R(Y, X)U)] \\ + \frac{\lambda}{(n-1)} [\eta(R(Y, Z)U) - g(X, R(Y, Z)U) - \eta(Y)\eta(R(\xi, Z)U) \\ + g(X, Y)\eta(R(\xi, Z)U) + g(X, Z)\eta(R(Y, \xi)U) - \eta(Z)\eta(R(Y, X)U) \\ + g(X, Y)\eta(R(Y, Z)\xi) - \eta(U)\eta(R(Y, Z)\xi)] \\ - \frac{1}{(n-1)} [g(\varphi X, R(Y, Z)U) - g(\varphi X, Y)\eta(R(\xi, Z)U) \\ - g(\varphi X, Z)\eta(R(Y, \xi)U) - g(\varphi X, U)\eta(R(Y, Z)\xi)] = 0.$$

By using (2.7) and (2.14) in (5.3), we have

$$(5.4) \quad \left[\frac{\lambda}{(n-1)} - 1 \right] [g(Y, U) - \eta(U)\eta(Y)] = 0.$$

Since $g(Y, U) \neq \eta(U)\eta(Y)$ therefore we have

$$(5.5) \quad \lambda = (n - 1).$$

Hence we have the following theorem:

Theorem 5.1. *Ricci soliton in para-Sasakian manifolds satisfying the condition $P(\xi, X).R = 0$ is always expanding.*

6. Ricci soliton in para-Sasakian manifolds Satisfying $S(\xi, X).P = 0$

In this section, we take $S(\xi, X).P = 0$, then we have

$$(6.1) \quad \begin{aligned} (S(X, \xi).P)(Y, Z)U &= ((X \wedge_S \xi)P)(Y, Z)U \\ &= (X \wedge_S \xi)P(Y, Z)U + P((X \wedge_S \xi)(Y, Z)U) \\ &\quad + P(Y, (X \wedge_S \xi)Z)U + P(Y, Z)(X \wedge_S \xi)U, \end{aligned}$$

where the endomorphism $X \wedge_S Y$ is defined by

$$(6.2) \quad (X \wedge_S Y)Z = S(Y, Z)X - S(X, Z)Y.$$

From (6.1) and (6.2), we have

$$(6.3) \quad \begin{aligned} (S(X, \xi).P)(Y, Z)U &= S(\xi, P(Y, Z)U)X - S(X, P(Y, Z)U)\xi \\ &\quad + S(\xi, Y)P(X, Z)U - S(X, Y)P(\xi, Z)U + S(\xi, Z)P(Y, X)U \\ &\quad - S(X, Z)P(Y, \xi)U + S(\xi, U)P(Y, Z)X - S(X, U)P(Y, Z)\xi. \end{aligned}$$

Putting $(S(X, \xi).P)(Y, Z)U = 0$ in (6.3), we have

$$(6.4) \quad \begin{aligned} S(\xi, P(Y, Z)U)X - S(X, P(Y, Z)U)\xi + S(\xi, Y)P(X, Z)U \\ - S(X, Y)P(\xi, Z)U + S(\xi, Z)P(Y, X)U - S(X, Z)P(Y, \xi)U \\ + S(\xi, U)P(Y, Z)X - S(X, U)P(Y, Z)\xi = 0. \end{aligned}$$

Taking the inner product with ξ in (6.4) and using (2.7), (2.8) we have

$$(6.5) \quad \begin{aligned} S(\xi, P(Y, Z)U)\eta(X) - S(X, P(Y, Z)U) + S(\xi, Y)\eta(P(X, Z)U) \\ - S(X, Y)\eta(P(\xi, Z)U) + S(\xi, Z)\eta(P(Y, X)U) - S(X, Z)\eta(P(Y, \xi)U) \\ + S(\xi, U)\eta(P(Y, Z)X) - S(X, U)\eta(P(Y, Z)\xi) = 0. \end{aligned}$$

By Using (2.20) and (2.27) in (6.5), we have

$$(6.6) \quad \begin{aligned} S(\xi, P(Y, Z)U) &= \left[\lambda - \frac{1}{(n-1)} \right] [-2\lambda(g(Z, U)\eta(Y) - g(Y, U)\eta(Z)) \\ &\quad + (S(\xi, Y)\eta(Z) - S(Z, \xi)\eta(Y))\eta(U) - \lambda g(Z, \xi)\eta(Y)\eta(U) \\ &\quad + S(\xi, Z)g(Y, U) - S(\xi, Y)g(Z, U)] \\ &\quad - \frac{1}{(n-1)} [-2g(\varphi Z, U)\eta(Y) + 2\lambda g(\varphi Y, W)\eta(Z) + S(\xi, Z)g(\varphi Y, U)]. \end{aligned}$$

Replacing Y by ξ in (6.6) and using (2.7), (2.8), we have

$$(6.7) \quad -\eta(P(\xi, Z)U) = \left[\lambda - \frac{1}{(n-1)} \right] [-2\lambda(g(Z, U) - \eta(U)\eta(Z)) + \lambda\eta(Z)]$$

$$-\frac{1}{(n-1)}[-2\lambda g(\varphi Z, U) + \lambda g(\varphi Z, U) - \lambda \eta(\varphi Z)\eta(U)].$$

By using (2.25) in (6.7), we have

$$(6.8) \quad \left[\lambda - \frac{1}{(n-1)} \right] [-\lambda g(Z, U) + \lambda \eta(U) \eta(Z) + \lambda \eta(Z)] = 0.$$

Taking $Z = U = \xi$ in (6.8) and (2.7), (2.8), we have

$$(6.9) \quad \left[\lambda - \frac{1}{(n-1)} \right] \lambda = 0.$$

Hence we can leads to the following theorem:

Theorem 6.1. *Ricci soliton in para-Sasakian manifolds satisfying the condition $S(\xi, X).P = 0$ is either steady or expanding.*

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