

A Study of Certain New Connections in an Almost Complex Manifold -II

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(Received January 10, 2024, Accepted February 22, 2024)

Abstract: In this paper we have defined several new connections and obtained curvature properties based on curvature tensors related with those coefficients of connections. Some interesting results related with those connection and curvature tensors in almost complex manifold have been obtained.

Keywords: Complex space, Symmetric and Skew symmetric tensor, Kahler space, Ricci tensor, Bianchi- identities.

1. Introduction

Let (M^n, g) be an even dimensional space of class C^{r+1} , with co-ordinate neighbourhoods (U, Σ^h) , $(h = 1, 2, 3 \dots n)$ and let g_{ij} and F_i^h be respectively the components of the Hermitian metric tensor and an almost complex structure tensor of M^n , such that

$$(1.1) \quad F_j^i F_i^h = -\delta_j^h$$

and

$$(1.2) \quad F_j^t F_i^s g_{ts} = g_{ij}.$$

If $F_{ji} = F_j^i g_{ti}$ then we can observe that $F_{ji} = -F_{ij}$ and $F_j^t F_{ht} = g_{jh}$. Let K_{kjih} , K_{ji} and K be respectively the curvature tensor, Ricci tensor and curvature scalar of M^n , we have¹ :

$$(1.3) \quad H_{kj} = (1/2)K_{kjih}F^{ih},$$

$$(1.4) \quad H_{hj} = K_{hs}F_j^s, \quad K_{kj} = -H_{ks}F_j^s$$

and

$$(1.5) \quad K = -H_{kj}F^{kj}.$$

Let ∇ be a linear connection in a Riemannian manifold M^n of dimension $n(> 2)$, then the quarter-symmetric connection is defined and expressed locally by Golab² as follows:

$$(1.6) \quad \Gamma_{ji}^h = \left\{ \begin{matrix} h \\ ji \end{matrix} \right\} + T_j^h p_i - T_{ji} p^h$$

where $\left\{ \begin{matrix} h \\ ji \end{matrix} \right\}$ are christoffel symbols, T_{ji} is a symmetric tensor and p_i is a 1-form such that $p^h = g^{ij}p_i$.

Various properties of quarter-symmetric connections have been studied by several authors including Dwivedi *et al.*³, Rastogi⁴⁻⁶, Mishra and Pandey⁷, Yano¹, Yano and Imai⁸, in which they gave the definition of the most general form of quarter-symmetric connection as follows:

$$(1.7) \quad \Gamma_{ji}^h = \left\{ \begin{matrix} h \\ ji \end{matrix} \right\} + U_j^h p_i - p_j V_i^h - U_{ji} p^h,$$

where U_{ji} and V_{ji} are respectively symmetric and skew-symmetric parts of an arbitrary tensor T_{ji} and $U_j^h = U_{jt} g^{ht}$, $V_j^h = V_{jt} g^{ht}$.

The connection Γ_{ji}^h given by equation (1.7) is such that the torsion tensor is given by $T_j^h p_i - T_i^h p_j$ and curvature tensor R_{kji}^h is expressible as

$$(1.8) \quad R_{kji}^h = K_{kji}^h - \varsigma_{(j,k)}[U_k^h p_{ji} + p_k^h U_{ji} - p_i \nabla_k U_j^h + p^h \nabla_k U_{ji} + V_i^h \nabla_k p_j \\ + p_j V_i^h \nabla_k - V_t^h p_j p_i U_k^t + U_{jt} V_i^t p_k p^h + U_k^h V_i^t p_j p_t - p_j p^t V_t^h U_{ji}]$$

where $p_{ji} = \nabla_j p_i - U_{jt} p^t p_i + (1/2)U_{ji} p_t p^t$, $p_i^h = p_{it} g^{th}$ and $\varsigma_{(j,k)}$ means interchange of indices j & k and subtraction.

2. First Connection Parameter $\Gamma_{ji}^{(1)h}$

Let us assume that the Christoffel symbol is being transformed to a connection $\Gamma_{ji}^{(1)h}$ given by equation (2.1) in the following form:

$$(2.1) \quad \Gamma_{ji}^{(1)h} = \left\{ \begin{matrix} h \\ ji \end{matrix} \right\} - F_{ji} p^h,$$

where p^h is a non-zero vector field.

Let $\nabla_m^{(1)}$ be the covariant derivative based on coefficient of connection $\Gamma_{ji}^{(1)h}$, then from equation (2.1), for a tensor T_{bc}^a , we have

$$(2.2) \quad \nabla_m^{(1)} T_{bc}^a = \nabla_m T_{bc}^a - F_{mr} T_{bc}^r p^a + p^r (F_{mb} T_{rc}^a + F_{mc} T_{br}^a),$$

which easily yields

$$(2.3)a \quad \nabla_m^{(1)} g_{bc} = F_{mb} p_c + F_{mc} p_b$$

and

$$(2.3)b \quad \nabla_m^{(1)} F_b^a = \nabla_m F_b^a - g_{mb} p^a + F_{mb} F_r^a p^r.$$

Hence we have

Theorem 2.1: *The connection $\Gamma_{ji}^{(1)h}$ is a non-metric connection as well as the covariant derivative of the tensor F_b^a satisfies equation (2.3)b.*

If $S_{ji}^{(1)h}$ is the torsion tensor of connection $\Gamma_{ji}^{(1)h}$, from equation (2.1), we can obtain

$$(2.4) \quad S_{ji}^{(1)h} = \Gamma_{ji}^{(1)h} - \Gamma_{ij}^{(1)h} = 2F_{ij} p^h.$$

Multiplying equation (2.1) by F_t^i , we get after some simple calculation

$$(2.5) \quad p^h = F_a^t g^{rh} \left(\left\{ \begin{matrix} a \\ rt \end{matrix} \right\} - \Gamma_{rt}^{(1)a} \right),$$

which leads to

$$(2.6) \quad S_{ji}^{(1)h} = F_a^t g^{rh} \left(\left\{ \begin{matrix} a \\ rt \end{matrix} \right\} - \Gamma_{rt}^{(1)a} \right).$$

From equation (2.1) and (2.5) we obtain and define following connection parameter

$$(2.7) \quad {}^* \Gamma_{ji}^{(1)h} = \Gamma_{ji}^{(1)h} - F_{ji} F_a^t g^{rh} \Gamma_{rt}^{(1)a} = \left\{ \begin{matrix} h \\ ji \end{matrix} \right\} - F_{ji} F_a^t g^{rh} \left\{ \begin{matrix} a \\ rt \end{matrix} \right\},$$

which gives the following theorem:

Theorem 2.2: *The connection parameter ${}^* \Gamma_{ji}^{(1)h}$, defined by equation (2.7), is an invariant connection parameter obtained by the transformation of Christoffel symbol with the help of equation (2.1).*

If ${}^* \nabla_m^{(1)}$ is the covariant derivative based on coefficient of connection ${}^* \Gamma_{ji}^{(1)h}$, we get

$$(2.8)a \quad {}^* \nabla_m^{(1)} g_{bc} = F_a^t (F_{mc} \left\{ \begin{matrix} a \\ bt \end{matrix} \right\} + F_{mb} \left\{ \begin{matrix} a \\ ct \end{matrix} \right\})$$

and

$$(2.8)b \quad {}^* \nabla_m^{(1)} F_b^a = \nabla_m F_b^a - F_s^t \left\{ \begin{matrix} s \\ ut \end{matrix} \right\} (g_{mb} g^{au} - F_{mb} F^{au})$$

Hence, we have:

Theorem 2.3: The covariant derivative ${}^*\nabla_m^{(1)}$, based on coefficient of connection ${}^*\Gamma_{ji}^{(1)h}$ is non-metric and satisfies equation (2.8)b.

3. Curvature Tensor $R_{kji}^{(1)h}$

Let $R_{kji}^{(1)h}$ be the curvature tensor based on coefficient of connection $\Gamma_{kji}^{(1)h}$, then we obtain

$$(3.1) \quad R_{kji}^{(1)h} = K_{kji}^h - \zeta_{(j,k)} [p^h \nabla_k F_{ji} + F_{ji} p^h \nabla_k - p^a p^h F_{ka} F_{ji}]$$

which leads to

$$(3.2) \quad R_{ji}^{(1)} = K_{ji} - \nabla_h (F_{ji} p^h) + \nabla_j (F_{hi} p^h) - p^a p^h F_{ja} F_{hi}$$

and

$$(3.3) \quad R^{(1)} = K + p^2 + \nabla_j (F_h^j p^h).$$

From equation (3.1) we can obtain following curvature identity

$$(3.4) \quad [R_{kji}^{(1)h} + 2\nabla_k (p^h F_{ij}) + 2p^a p^h F_{ka} p^h F_{ji}] + \text{Cycl}(i, j, k) = 0,$$

while equation (3.2) gives

$$(3.5) \quad R_{ji}^{(1)} - R_{ij}^{(1)} = \nabla_j (F_{hi} p^h) - \nabla_i (F_{hj} p^h) - 2\nabla_h (F_{ji} p^h).$$

From equations (3.3), (3.4) and (3.5), we have:

Theorem 3.1: (i) The curvature tensor $R_{kji}^{(1)h}$, obtained from the connection coefficients $\Gamma_{ji}^{(1)h}$ satisfies curvature identity (3.4).

(ii) The Ricci curvature $R_{ji}^{(1)}$ shall be symmetric in i & j if and only if

$$\nabla_j (F_{hi} p^h) - \nabla_i (F_{hj} p^h) = 2\nabla_h F_{ji} p^h,$$

$$(iii) \quad R^{(1)} = K + p^2 + \nabla_j F_h^j p^h.$$

In case of Kahler space using $\nabla_j (F_{hi}) = 0$, we can obtain

Theorem 3.2: In case of Kahler space,

(i) The curvature tensor $R_{kji}^{(1)h}$, satisfies identity

$$[R_{kji}^{(1)h} + 2F_{ji} \nabla_k p^h + 2p^a p^h F_{ka} F_{ji}] + \text{Cycl.}(i, j, k) = 0,$$

(ii) The Ricci tensor $R_{ji}^{(1)}$ shall be symmetric in i & j if and only if

$$2F_{ji} \nabla_h p^h = F_{hi} \nabla_j p^h - F_{hj} \nabla_i p^h,$$

$$(iii) \quad R^{(1)} = K + p^2 + F_h^j \nabla_j p^h.$$

From equation (3.1), we can obtain

$$(3.6) \quad \nabla_m R_{kji}^{(1)} = \nabla_m K_{kjih} - \varsigma_{(j,k)} [(\nabla_m p^h)(\nabla_k F_{ji} - p^a F_{ka} F_{ji}) + (\nabla_m F_{ji})(\nabla_k p_h) \\ + p_h \{(\nabla_m \nabla_k F_{ji}) - p^a (\nabla_m F_{ka}) F_{ji} - p^a (\nabla_m F_{ji}) F_{ka}\} \\ + F_{ji} \{(\nabla_m \nabla_k p_h) - (\nabla_m p^a) p_h F_{ka}\}]$$

which leads to following curvature identity

$$(3.7) \quad [\nabla_m R_{kji}^{(1)} + [p^a (\nabla_m p_h)(F_{ja} F_{ki} - F_{ka} F_{ji} - p^a p_h \{(\nabla_m F_{ka} - \nabla_k F_{ma}) F_{ji} \\ + (\nabla_m F_{ji} - \nabla_j F_{mi}) F_{ka}\} + p_h (\nabla_m \nabla_k - \nabla_k \nabla_m) F_{ji} + F_{ji} (\nabla_m \nabla_k - \nabla_k \nabla_m) p_h \\ - F_{ji} p_h \{(\nabla_m p^a) F_{ka} - (\nabla_k p^a) F_{ka} - (\nabla_k p^a) F_{ma}\}]] + Cycl. (i, j, k) = 0.$$

In case of Kahler space equations (3.6) and (3.7) respectively reduce to

$$(3.8) \quad \nabla_m R_{kji}^{(1)} = \nabla_m K_{kjih} - \varsigma_{(j,k)} [(\nabla_m p_h) p^a F_{ak} F_{ji} + F_{ji} \{(\nabla_m \nabla_k p_h) \\ + (\nabla_m p^a) p_h F_{ak}\}]$$

and

$$(3.9) \quad [\nabla_m R_{kji}^{(1)} + [p^a (\nabla_m p_h)(F_{ja} F_{ki} - F_{ka} F_{ji}) + F_{ji} (\nabla_m \nabla_k - \nabla_k \nabla_m) p_h \\ - F_{ji} p_h \{(\nabla_m p^a) F_{ka} - (\nabla_k p^a) F_{ma}\}]] + Cycl. (j, m, k) = 0.$$

From equation (3.8) and (3.9), we have the following theorem:

Theorem 3.3: In case of Kahler space,

(i) Covariant derivative of the curvature tensor $R_{kji}^{(1)}$, satisfies equation (3.8),

(ii) The Ricci tensor $R_{ji}^{(1)}$ shall be symmetric in i & j if and only if

$$\nabla_m R_{kji}^{(1)} = \nabla_m K_{kjih},$$

(iii) The Bianchi identity based on curvature tensor $R_{kji}^{(1)}$, is given by equation (3.9).

4. Curvature Tensor $*R_{kji}^{(1)h}$

Let $*R_{kji}^{(1)h}$ be the curvature tensor based on coefficients of connection $*\Gamma_{ji}^{(1)h}$, then we can obtain by virtue of equation (2.7)

$$(4.1) \quad *R_{kji}^{(1)h} = K_{kji}^h + \varsigma_{(j,k)} \left[g^{rh} \left\{ \begin{smallmatrix} u \\ rt \end{smallmatrix} \right\} F_u^t F_{ji} + g^{rh} \left\{ \begin{smallmatrix} a \\ rt \end{smallmatrix} \right\} F_{ki} \partial_j F_a^t + \right. \\ \left. F_a^t \left\{ \begin{smallmatrix} a \\ rt \end{smallmatrix} \right\} F_{ki} \partial_j g^{rh} + F_a^t g^{rh} F_{ki} \partial_j \left\{ \begin{smallmatrix} a \\ rt \end{smallmatrix} \right\} + \right. \\ \left. F_v^s g^{pa} \left\{ \begin{smallmatrix} v \\ ps \end{smallmatrix} \right\} F_{ki} \left\{ \begin{smallmatrix} h \\ ja \end{smallmatrix} \right\} + F_u^t F_v^s g^{rh} \left\{ \begin{smallmatrix} u \\ rt \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} v \\ ps \end{smallmatrix} \right\} F_{ji} F_k^p \right].$$

From equation (3.1) and (4.1) we obtain a relation between curvature tensor $*R_{kji}^{(1)h}$ and $R_{kji}^{(1)h}$ in the following form:

$$(4.2) \quad *R_{kji}^{(1)h} = R_{kji}^{(1)h} + \varsigma_{(j,k)} \left[\left(p^h + g^{rh} \left\{ \begin{smallmatrix} u \\ rt \end{smallmatrix} \right\} F_u^t \right) (\nabla_k F_{ji}) + F_{ji} \left(\nabla_k p^h - g^{rh} \left\{ \begin{smallmatrix} a \\ rt \end{smallmatrix} \right\} \partial_k F_a^t - F_a^t \left\{ \begin{smallmatrix} a \\ rt \end{smallmatrix} \right\} \partial_k g^{rh} - F_a^t g^{rh} \partial_k \left\{ \begin{smallmatrix} a \\ rt \end{smallmatrix} \right\} + F_v^s g^{pa} \left\{ \begin{smallmatrix} v \\ ps \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} h \\ ka \end{smallmatrix} \right\} + F_u^t F_v^s g^{rh} \left\{ \begin{smallmatrix} u \\ rt \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} v \\ ps \end{smallmatrix} \right\} F_k^p - p^a p^h F_{ka} \right) \right].$$

From equation (4.1), we can obtain on contraction for k & h

$$(4.3) \quad *R_{ji}^{(1)} = K_{ji} + g^{rh} \left\{ \begin{smallmatrix} u \\ rt \end{smallmatrix} \right\} F_u^t (\nabla_h F_{ij} - \nabla_j F_{hi}) + g^{rh} \left\{ \begin{smallmatrix} u \\ rt \end{smallmatrix} \right\} F_u^t (F_{hi} \partial_j F_u^t - F_{ji} \partial_h F_u^t) + F_u^t \left\{ \begin{smallmatrix} u \\ rt \end{smallmatrix} \right\} (F_{hi} \partial_j g^{rh} - F_{ji} \partial_h g^{rh}) + F_u^t g^{rh} (F_{hi} \partial_j \left\{ \begin{smallmatrix} u \\ rt \end{smallmatrix} \right\} - F_{ji} \partial_h \left\{ \begin{smallmatrix} u \\ rt \end{smallmatrix} \right\}) + F_u^t F_v^s g^{rh} \left\{ \begin{smallmatrix} u \\ rt \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} v \\ ps \end{smallmatrix} \right\} \cdot (F_{ji} F_h^p - F_{hi} F_j^p),$$

which shows that

$$(4.4) \quad *R_{ji}^{(1)} - *R_{ij}^{(1)} = g^{rh} \left\{ \begin{smallmatrix} u \\ rt \end{smallmatrix} \right\} F_u^t (2\nabla_h F_{ji} - \nabla_j F_{hi} - \nabla_i F_{hj}) + g^{rh} \left\{ \begin{smallmatrix} u \\ rt \end{smallmatrix} \right\} (F_{hi} \partial_j F_u^t - F_{hj} \partial_i F_u^t - 2F_{ji} \partial_h F_u^t) + F_u^t \left\{ \begin{smallmatrix} u \\ rt \end{smallmatrix} \right\} (F_{hi} \partial_j g^{rh} - F_{hj} \partial_i g^{rh} - 2F_{ji} \partial_h g^{rh}) + F_v^s g^{pa} \left\{ \begin{smallmatrix} v \\ ps \end{smallmatrix} \right\} \left(F_{hi} \left\{ \begin{smallmatrix} h \\ ja \end{smallmatrix} \right\} - F_{hj} \left\{ \begin{smallmatrix} h \\ ia \end{smallmatrix} \right\} - 2F_{ji} \left\{ \begin{smallmatrix} h \\ ha \end{smallmatrix} \right\} \right) + (F_u^t F_v^s g^{rh} \left\{ \begin{smallmatrix} u \\ rt \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} v \\ ps \end{smallmatrix} \right\}) \cdot (2F_{ji} F_h^p - F_{hi} F_j^p - F_{hj} F_i^p),$$

Further from equation (4.3) we can obtain

$$(4.5) \quad R^{(1)} = K - g^{rh} \left\{ \begin{smallmatrix} u \\ rt \end{smallmatrix} \right\} (F_u^t \nabla_j F_h^j - F_h^j \partial_j F_u^t) + F_u^t F_h^j \left\{ \begin{smallmatrix} u \\ rt \end{smallmatrix} \right\} \partial_j g^{rh} + g^{rh} \partial_j \left\{ \begin{smallmatrix} u \\ rt \end{smallmatrix} \right\} + 2F_v^s g^{pa} F_h^j \left\{ \begin{smallmatrix} v \\ ps \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} h \\ ja \end{smallmatrix} \right\}.$$

In case of Kahler space equation (4.2), (4.4) and (4.5) respectively give

$$(4.6)a \quad *R_{kji}^{(1)h} = R_{kji}^{(1)h} + \varsigma_{(j,k)} [F_{ji} (\nabla_k p^h - g^{rh} \left\{ \begin{smallmatrix} a \\ rt \end{smallmatrix} \right\} \partial_k F_a^t - F_a^t \left\{ \begin{smallmatrix} a \\ rt \end{smallmatrix} \right\} \partial_k g^{rh} - F_a^t g^{rh} \partial_k \left\{ \begin{smallmatrix} a \\ rt \end{smallmatrix} \right\} + F_v^s g^{pa} \left\{ \begin{smallmatrix} v \\ ps \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} h \\ ka \end{smallmatrix} \right\} + F_u^t F_v^s g^{rh} \left\{ \begin{smallmatrix} u \\ rt \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} v \\ ps \end{smallmatrix} \right\} F_k - p^a p^h F_{ka})]$$

$$\begin{aligned}
(4.6)b \quad *R_{ji}^{(1)} - *R_{ij}^{(1)} = & g^{rh} \left\{ \begin{matrix} u \\ rt \end{matrix} \right\} (F_{hi} \partial_j F_u^t - F_{hj} \partial_i F_u^t - 2F_{ji} \partial_h F_u^t) \\
& + F_u^t \left\{ \begin{matrix} u \\ rt \end{matrix} \right\} (F_{hi} \partial_j g^{rh} - F_{hj} \partial_i g^{rh} - 2F_{ji} \partial_h g^{rh}) \\
& + F_u^t g^{rh} \left(F_{hi} \partial_j \left\{ \begin{matrix} u \\ rt \end{matrix} \right\} - F_{hj} \partial_i \left\{ \begin{matrix} u \\ rt \end{matrix} \right\} - 2F_{ji} \partial_h \left\{ \begin{matrix} u \\ rt \end{matrix} \right\} \right) \\
& + F_v^s g^{pa} \left\{ \begin{matrix} v \\ ps \end{matrix} \right\} \left(F_{hi} \left\{ \begin{matrix} h \\ ja \end{matrix} \right\} - F_{hj} \left\{ \begin{matrix} h \\ ia \end{matrix} \right\} - 2F_{ji} \left\{ \begin{matrix} h \\ ha \end{matrix} \right\} \right) \\
& + (F_u^t F_v^s g^{rh} \left\{ \begin{matrix} u \\ rt \end{matrix} \right\} \left\{ \begin{matrix} v \\ ps \end{matrix} \right\}) \cdot (2F_{ji} F_h^p - F_{hi} F_j^p - F_{hj} F_i^p),
\end{aligned}$$

and

$$\begin{aligned}
(4.6)c \quad *R^{(1)} = & K + g^{rh} F_u^t F_j^h \partial_j \left\{ \begin{matrix} u \\ rt \end{matrix} \right\} + 2g^{pa} F_h^j F_s^v \left\{ \begin{matrix} v \\ ps \end{matrix} \right\} \left\{ \begin{matrix} h \\ ja \end{matrix} \right\} \\
& + \left\{ \begin{matrix} u \\ rt \end{matrix} \right\} [g^{rh} F_h^j F_u^p \left\{ \begin{matrix} t \\ pj \end{matrix} \right\} - g^{ph} \left\{ \begin{matrix} r \\ pj \end{matrix} \right\} F_u^t F_h^j] \\
& + g^{rp} (F_p^j F_h^t \left\{ \begin{matrix} h \\ ju \end{matrix} \right\} - F_u^t F_h^j \left\{ \begin{matrix} h \\ pj \end{matrix} \right\})
\end{aligned}$$

Hence, we have:

Theorem 4.1: In case of a Kahler space tensors $*R_{kji}^{(1)h}$, $*R_{ji}^{(1)}$ and $*R^{(1)}$ satisfy equations (4.6)a, (4.6)b and (4.6)c.

5. Second Connection Parameter $\Gamma_{ji}^{(2)h}$

Let us assume that the Christoffel symbol is being transformed to a connection $\Gamma_{ji}^{(2)h}$ given by equation (5.1) in the following form:

$$(5.1) \quad \Gamma_{ji}^{(2)h} = \left\{ \begin{matrix} h \\ ji \end{matrix} \right\} - g_{ij} q^h - q_j F_i^h,$$

where q^h is a non-zero vector field. Let $\nabla_m^{(2)}$ be the covariant derivative based on coefficient of connection $\Gamma_{ji}^{(2)h}$, then from equation (5.1), for tensor T_{bc}^a , we can obtain

$$\begin{aligned}
(5.2) \quad \nabla_m^{(2)} T_{bc}^a = & \nabla_m T_{bc}^a - q^a g_{mr} T_{bc}^r + q^a (g_{mb} T_{rc}^a + g_{mc} T_{br}^a) \\
& - q_m (F_r^a T_{bc}^r - F_b^r T_{rc}^a - F_c^r T_{br}^a),
\end{aligned}$$

which leads to

$$(5.3)a \quad \nabla_m^{(2)} g_{bc} = q_b g_{mc} + q_c g_{mb}$$

and

$$(5.3)b \quad \nabla_m^{(2)} F_b^a = \nabla_m F_b^a - q^a F_{bm} + q^r F_r^a g_{mb}.$$

Hence, we have:

Theorem 5.1: *The connection $\Gamma_{ji}^{(2)h}$ is a non- metric connection as well as the covariant derivative of the tensor F_b^a satisfies equation (5.3)b.*

In case of Kahler space, equation (5.3) b reduces to

$$(5.4) \quad \nabla_m^{(2)} F_b^a + q^a F_{bm} - q^r F_r^a g_{mb} = 0.$$

If $S_{ji}^{(2)h}$ is a torsion tensor of connection $\Gamma_{ji}^{(2)h}$, from equation (5.1), we can obtain

$$(5.5) \quad S_{ji}^{(2)h} = \Gamma_{ji}^{(2)h} - \Gamma_{ij}^{(2)h} = q_i F_j^h - q_j F_i^h.$$

Contracting equation (5.1) for h & i , we have

$$(5.6) \quad q_j = \left\{ \begin{matrix} h \\ jh \end{matrix} \right\} - \Gamma_{ji}^{(2)h}.$$

Substituting the value of q_i in equation (5.1), we get

$$(5.7) \quad {}^*\Gamma_{ji}^{(2)h} = \Gamma_{ji}^{(2)h} - F_i^h \Gamma_{ja}^{(2)a} - g_{ji} g^{rh} \Gamma_{ra}^{(2)a} = \left\{ \begin{matrix} h \\ ji \end{matrix} \right\} - F_i^h \left\{ \begin{matrix} a \\ ja \end{matrix} \right\} - g_{ji} g^{rh} \left\{ \begin{matrix} a \\ ra \end{matrix} \right\}.$$

Hence, we have:

Theorem 5.2: *The connection parameter ${}^*\Gamma_{ji}^{(2)h}$, defined by equation (5.7), is an invariant connection parameter obtained by the transformation of Christoffel symbol with the help of equation (5.1).*

If ${}^*\nabla_m^{(2)}$ is the covariant derivative based on coefficient of connection ${}^*\Gamma_{ji}^{(2)h}$, we can obtain

$$(5.8) \quad {}^*\nabla_m^{(2)} T_{bc}^a = \nabla_m T_{bc}^a - F_a^m \left\{ \begin{matrix} d \\ rd \end{matrix} \right\} T_{bc}^r - g_{mr} g^{pa} \left\{ \begin{matrix} d \\ pd \end{matrix} \right\} T_{bc}^r \\ + g^{pr} \left\{ \begin{matrix} d \\ pd \end{matrix} \right\} (g_{mb} T_{rc}^a + g_{mc} T_{rb}^a) + F_m^r \left(\left\{ \begin{matrix} d \\ bd \end{matrix} \right\} T_{rc}^a + \left\{ \begin{matrix} d \\ cd \end{matrix} \right\} T_{br}^a \right),$$

which gives

$$(5.9)a \quad {}^*\nabla_m^{(2)} g_{bc} = \left\{ \begin{matrix} d \\ bd \end{matrix} \right\} (g_{mc} + F_{mc}) + \left\{ \begin{matrix} d \\ cd \end{matrix} \right\} (g_{mb} + F_{mb})$$

and

$$(5.9)b \quad {}^*\nabla_m^{(2)} F_b^a = \nabla_m F_b^a - \left\{ \begin{matrix} d \\ pd \end{matrix} \right\} (F_{bm} + F_{mc} g^{pa} - F^{pa} g_{mb}).$$

Hence, we have:

Theorem 5.3: *The covariant derivative ${}^*\nabla_m^{(2)}$, based on coefficient of connection ${}^*\Gamma_{ji}^{(2)h}$ is a non- metric and satisfies equation (5.9) a and (5.9)b.*

6. Curvature Tensor $R_{ji}^{(2)h}$ and $*R_{kji}^{(2)h}$

Let $R_{ji}^{(2)h}$ be the curvature tensor based on coefficient of connection $R_{ji}^{(2)h}$, then we can obtain from equation (5.1), following expression for curvature tensor $R_{ji}^{(2)h}$:

$$(6.1) \quad R_{ji}^{(2)h} = K_{ji}^h - \varsigma_{(j,k)}[g_{ji}\nabla_k q^h - q_k\nabla_j F_i^h - q^h\partial_k g_{ji} - q^h q_j F_{ik} - F_h^a q^a q_k g_{ji}].$$

From equation (6.1), we can obtain on simplification

$$(6.2) \quad R_{ji}^{(2)} = K_{ji} - [g_{ji}(\nabla_h q^h - q^2 - F_h^a q^a q_h) - (\nabla_j q_i - q_k q_i) + q^2 F_{ij} - 2q^h q_j F_{ih} - q_h \nabla_j F_i^h + q_j \nabla_h F_i^h]$$

and

$$(6.3) \quad R^{(2)} = K - [(n-1)(\nabla_h q^h - q^2) - nF_a^h q^a q_h].$$

From equation (6.1), we can obtain following curvature identity

$$(6.4) \quad R_{kji}^{(2)h} + q_i(\nabla_j F_k^h - \nabla_k F_j^h) + 2q^h q_j F_{ij} + Cycl.(i, j, k) = 0,$$

while equation (6.2) gives

$$(6.5) \quad \varsigma_{(i,j)}[R_{ji}^{(2)} + \nabla_i q_j + q^2 F_{ij} + 2q^h q_i F_{jh} + q_h \nabla_i F_j^h + q_j \nabla_h F_i^h] = 0.$$

From Equations (6.3), (6.4) and (6.5), we have the following theorem:

Theorem 6.1: (i) The curvature tensor $R_{kji}^{(2)h}$, obtained from the connection coefficients $\Gamma_{ji}^{(2)h}$ satisfies curvature identity (6.4),

(ii) The Ricci tensor $R_{ji}^{(2)}$ shall be symmetric in i & j if and only if

$$\varsigma_{(i,j)}[\nabla_i q_j + q^2 F_{ij} + 2q^h q_i F_{jh} + q_h \nabla_i F_j^h + q_j \nabla_h F_i^h = 0],$$

$$(iii) \quad R^{(2)} = K - [(n-1)(\nabla_h q^h - q^2) - nF_a^h q^a q_h].$$

In case of Kahler space using $\nabla_j F_i^h = 0$, we get

Theorem 6.2: In case of Kahler space

(i) The curvature tensor $R_{kji}^{(2)h}$, satisfies identity

$$[R_{kji}^{(2)h} + 2q^h q_j F_{ij} + Cycl.(i, j, k)] = 0$$

(ii) The Ricci tensor $R_{ji}^{(2)}$ shall be symmetric in i & j if and only if

$$\varsigma_{(i,j)}[\nabla_i q_j + q^2 F_{ij} + 2q^h q_i F_{jh} = 0].$$

From equation (6.1), we can obtain

$$(6.6) \quad \nabla_m R_{kjih}^{(2)} = \nabla_m K_{kjih} - \varsigma_{(j,k)} [g_{ji} \{ (\nabla_m \nabla_k q_h) - q_h (\nabla_m q_k) - q_k (\nabla_m q_h) \\ - q^a q_k (\nabla_m F_{ah}) - q_k F_{ah} (\nabla_m q^a) - F_{ah} q^a (\nabla_m q_k) \} \\ - (\nabla_m q_k) (\nabla_j F_{ih}) - q_k (\nabla_m \nabla_j F_{ih}) - (\nabla_m \nabla_j F_{ih}) \\ - (\nabla_m q_h) (q_j F_{ik}) - q_h (\nabla_m q_j) F_{ik} + q_h q_k (\nabla_m F_{ij})],$$

In case of Kahler space, equation (6.6) reduces to

$$(6.7) \quad \nabla_m R_{kjih}^{(2)} = \nabla_m K_{kjih} - \varsigma_{(j,k)} [g_{ji} \{ (\nabla_m \nabla_k q_h) - q_h (\nabla_m q_k) - q_k (\nabla_m q_h) \\ - q_k F_{ah} (\nabla_m q^a) - F_{ah} q^a (\nabla_m q_k) \} - q_h (\nabla_m q_j) F_{ik}],$$

which leads to following curvature identity:

$$(6.8) \quad [\nabla_m R_{kjih}^{(2)} + \varsigma_{(m,k)} g_{ji} \{ \nabla_m \nabla_k q_h - q_h \nabla_m q_k - q_k \nabla_m q_h - F_{ah} q_k \nabla_m q^a \\ - F_{ah} q^a \nabla_m q_k \} - F_{ji} q_h \nabla_m q_k] + Cycl. (j, m, k) = 0.$$

From equations (6.7) and (6.8), we can obtain following theorem:

Theorem 6.3: (i) Covariant derivative of the curvature tensor $R_{kjih}^{(2)}$, satisfies equation (6.7),

(ii) If vector q_h is co-variantly constant, equation (6.7) implies

$$\nabla_m R_{kjih}^{(2)} = \nabla_m K_{kjih},$$

(iii) The Bianchi-identity based on curvature tensor $R_{kjih}^{(2)}$ is given by (6.8).

Let $*R_{kji}^{(2)h}$, be the curvature tensor based on coefficient of connection $*\Gamma_{ji}^{(2)h}$, then we can obtain by virtue of equation (5.6)

$$(6.9) \quad *R_{kji}^{(2)h} = K_{kji}^h - \varsigma_{(j,k)} [F_i^a \left\{ \begin{smallmatrix} p \\ jp \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} h \\ ka \end{smallmatrix} \right\} - g_{ka} g^{rh} \left\{ \begin{smallmatrix} t \\ rt \end{smallmatrix} \right\} + F_i^h \partial_k \left\{ \begin{smallmatrix} a \\ ja \end{smallmatrix} \right\} \\ + g_{ji} \left\{ \begin{smallmatrix} t \\ rt \end{smallmatrix} \right\} (g^{ra} \left\{ \begin{smallmatrix} h \\ ka \end{smallmatrix} \right\} - g^{rh} \left\{ \begin{smallmatrix} p \\ kp \end{smallmatrix} \right\} - F_a^h g^{ra} \left\{ \begin{smallmatrix} p \\ kp \end{smallmatrix} \right\}) \\ - g^{rh} \left\{ \begin{smallmatrix} a \\ ra \end{smallmatrix} \right\} (\partial_k g_{ij}) + g_{ji} \left(\left\{ \begin{smallmatrix} a \\ ra \end{smallmatrix} \right\} (\partial_k g^{rh}) + g^{rh} \partial_k \left\{ \begin{smallmatrix} a \\ ra \end{smallmatrix} \right\}) \right]]$$

Contracting equation (6.9) for h & k , on simplification we get

$$(6.10) \quad *R_{ji}^{(2)} = K_{ji} + F_i^a [\partial_j \left\{ \begin{smallmatrix} h \\ ah \end{smallmatrix} \right\} - \partial_a \left\{ \begin{smallmatrix} h \\ jh \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} p \\ hp \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} h \\ ja \end{smallmatrix} \right\} - g_{ja} g^{rh} \left\{ \begin{smallmatrix} t \\ rt \end{smallmatrix} \right\}] \\ + g_{hi} \left\{ \begin{smallmatrix} t \\ rt \end{smallmatrix} \right\} [\partial_j g^{rh} + g^{ra} \left\{ \begin{smallmatrix} h \\ ja \end{smallmatrix} \right\} - g^{ra} F_a^h \left\{ \begin{smallmatrix} p \\ jp \end{smallmatrix} \right\}] + \partial_j \left\{ \begin{smallmatrix} a \\ ia \end{smallmatrix} \right\} \\ - g_{ji} [\left\{ \begin{smallmatrix} t \\ rt \end{smallmatrix} \right\} (\partial_h g^{rh} - F_a^h \left\{ \begin{smallmatrix} p \\ hp \end{smallmatrix} \right\} g^{ra}) + g^{rh} \partial_h \left\{ \begin{smallmatrix} t \\ rt \end{smallmatrix} \right\}] \\ + g^{rh} \left\{ \begin{smallmatrix} t \\ rt \end{smallmatrix} \right\} (\partial_h g_{ji} - \partial_j g_{hi}) - \left\{ \begin{smallmatrix} t \\ it \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} p \\ jp \end{smallmatrix} \right\}.$$

From Equation (6.10), we can obtain

$$(6.11) \quad \begin{aligned} *R_{ji}^{(1)} - *R_{ij}^{(1)} = & \varsigma_{(i,j)} [F_i^a (\partial_j \left\{ \begin{smallmatrix} h \\ ah \end{smallmatrix} \right\} - \partial_a \left\{ \begin{smallmatrix} h \\ jh \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} p \\ hp \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} h \\ ja \end{smallmatrix} \right\} \\ & - g_{ja} g^{rh} \left\{ \begin{smallmatrix} t \\ rt \end{smallmatrix} \right\})] + g_{hi} \left\{ \begin{smallmatrix} t \\ rt \end{smallmatrix} \right\} [\partial_j g^{rh} + g^{ra} \left\{ \begin{smallmatrix} h \\ ja \end{smallmatrix} \right\} \\ & - g^{ra} F_a^h \left\{ \begin{smallmatrix} p \\ jp \end{smallmatrix} \right\}] + \partial_j \left\{ \begin{smallmatrix} a \\ ia \end{smallmatrix} \right\} + g^{rh} \left\{ \begin{smallmatrix} t \\ rt \end{smallmatrix} \right\} (\partial_i g_{hj})]. \end{aligned}$$

Remark: From equations (6.9), (6.10) and (6.11) we can obtain results similar to section 5.

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