

A Study of Generalized Hurwitz-Lerch Zeta Function of Two-variables and Three-variables and its terms of Extended Beta Function

Priyanka Srivastava and S. K. Raizada

Department of Mathematics and Statistics,
Dr. Rammanohar Lohia Avadh University Ayodhya, (U.P.), India
Email: priyanka9794840993@gmail.com ; skr.65@rediffmail.com

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Abstract: The main aim of this paper is to introduce a new extension of the generalized Hurwitz-Lerch Zeta functions of two variables [abbreviated as H-L-Z-F-2V] and three variables [abbreviated as H-L-Z-F-3V] involving the extended Beta function. We then systematically investigate its several interesting properties and related formulas as (for example) various integral representations, which provide certain new and known extensions of earlier corresponding results, a derivative formula, and Mellin-transform. We also consider some important special cases.

Keywords: Generalized Hurwitz-Lerch Zeta Function, Beta Function, Gamma Function, Hypergeometric Function, Eulerian Integral.

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1. Introduction

Special functions of mathematical physics are considered as the solutions of Laplace equations, partial differential equations, diffusion equations and many other differential equations. Such Special functions are widely and usefully applied by scientists and engineers in various scientific and engineering problems. Recently much work has been done on the generalizations and extensions of previously established Special functions. A sufficient amount of work has also been done by various researchers like¹⁻¹¹ on generalized H-L-Z-F. motivated by the recent works on H-L-Z-F, we in the present paper have further studied these functions using extended Beta functions. A lot of research work has recently come up on the study and development of functions, that are more general than the Beta type function, properly known as generalized Beta type functions.

These functions as a part of the theory of confluent hypergeometric functions, are important Special functions as they are widely used in physics and engineering. Moreover, generalized Beta functions¹², have played a pivotal role in the advancement of further research

The generalized Hurwitz-Lerch Zeta function of two variables is defined by¹³

$$(1.1) \quad \Phi_{\mu,\eta}^*(z, t, s, a) = \sum_{k,l=0}^{\infty} \frac{(\mu)_k(\eta)_l}{(k+l+a)^s} \frac{z^k t^l}{k!l!}$$

$$(\mu, \eta \in \mathbb{C}; \quad a \in \mathbb{C} \setminus \mathbb{Z}_0^-; \quad s \in \mathbb{C} \text{ when } |z| < 1 \text{ and } |t| < 1;$$

$$\Re(s - \mu - \eta) > 0 \text{ when } |z| = 1 \text{ and } |t| = 1).$$

Recently Choi and Parmar¹⁴ introduced and studied the Hurwitz-Lerch Zeta function of two variables is defined as:

$$(1.2) \quad \Phi_{\mu,\eta,\eta';\nu}(z, t, s, a) = \sum_{k,l=0}^{\infty} \frac{(\mu)_{k+l}(\eta')_k(\eta'')_l}{(\nu)_{k+l}(k+l+a)^s} \frac{z^k t^l}{k!l!}$$

$$(\mu, \eta, \eta' \in \mathbb{C}; \quad a, \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-; \quad s, z, t \in \mathbb{C} \text{ when } |z| < 1 \text{ and } |t| < 1;$$

$$\text{and } \Re(s + \nu - \mu - \eta - \eta') > 0 \text{ when } |z| = 1 \text{ and } |t| = 1).$$

Very recently, the extended Hurwitz-Lerch Zeta function of two variables is defined by Nisar¹⁵ as:

$$(1.3) \quad \Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}(z, t, s, a) = \sum_{k,l=0}^{\infty} \frac{(\mu)_{k+l}(\eta)_k(\eta')_l(\delta)_k(\delta')_l}{(\nu)_{k+l}(\xi)_k(\xi')_l(k+l+a)^s} \frac{z^k t^l}{k!l!}$$

$$(\mu, \eta, \eta', \delta, \delta' \in \mathbb{C}; \quad a, \nu, \xi, \xi' \in \mathbb{C} \setminus \mathbb{Z}_0^-; \quad s, z, t \in \mathbb{C} \text{ when } |z| < 1 \text{ and } |t| < 1;$$

$$\text{and } \Re(s + \nu + \xi + \xi' - \mu - \eta - \eta' - \delta - \delta') > 0 \text{ when } |z| = 1 \text{ and } |t| = 1).$$

Hurwitz-Lerch Zeta function of two variables in (1.2) is generalized in three variables as:

$$(1.4) \quad \Phi_{\mu,\eta,\eta',\eta'';\nu}(z, t, r, s, a) = \sum_{k,l,m=0}^{\infty} \frac{(\mu)_{k+l}(\eta)_k(\eta')_l(\eta'')_m}{(\nu)_{k+l}(k+l+a)^s} \frac{z^k t^l r^m}{k!l!r!}$$

$$(\mu, \eta, \eta', \eta'' \in \mathbb{C}; \quad a, \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-; \quad s, z, t, r \in \mathbb{C} \text{ when } |z| < 1, |t| < 1 \text{ and } |r| < 1;$$

$$\text{and } \Re(s + \nu - \mu - \eta - \eta' - \eta'') > 0 \text{ when } |z| = 1, |t| = 1 \text{ and } |r| = 1).$$

Hurwitz-Lerch Zeta function of two variables in equation (1.3) is generalized in three variables as:

$$(1.5) \quad \Phi_{\mu,\eta,\eta',\eta'',\delta,\delta',\delta'';\nu,\xi,\xi',\xi''}(z, t, r, s, a)$$

$$= \sum_{k,l,m=0}^{\infty} \frac{(\eta)_k(\eta')_l(\eta'')_m(\delta)_k(\delta')_l(\delta'')_m}{(\xi)_k(\xi')_l(\xi'')_m(k+l+m+a)^s} \frac{z^k t^l r^m}{k!l!r!}$$

$(\mu, \eta, \eta', \eta'', \delta, \delta', \delta'' \in \mathbb{C}; \quad a, \nu, \xi, \xi', \xi'' \in \mathbb{C} \setminus \mathbb{Z}_0^-; \quad s, z, t, r \in \mathbb{C} \text{ when } |z| < 1, |t| < 1$
 $\text{and } |r| < 1; \text{ and } \Re(s + \nu + \xi + \xi' + \xi'' - \mu - \eta - \eta' - \eta'' - \delta - \delta' - \delta'') > 0$
 $\text{when } |z| = 1, |t| = 1 \text{ and } |r| = 1).$

2. Preliminaries

Generalization of extended Beta and extended Hypergeometric functions are given by Choi¹⁴:

$$(2.1) \quad B_{p,q}(\delta_1, \delta_2) = \int_0^1 t^{\delta_1-1} (1-t)^{\delta_2-1} \exp\left(-\frac{p}{t} - \frac{q}{(1-t)}\right) dt$$

$$(\Re(p) > 0, \Re(q) > 0, \Re(\delta_1) > 0, \Re(\delta_2) > 0),$$

$$(2.2) \quad F_{p,q}(a, b; c; \xi;) = \sum_{r=0}^{\infty} \frac{B_{p,q}(b+r, c-b)}{B(b, c-b)} (a)_m \frac{\xi^r}{r!}$$

$$(\Re(p) \geq 0, \Re(q) \geq 0, |\xi| < 1; \Re(c) > 0, \Re(b) > 0).$$

and

$$(2.3) \quad F_{p,q}(a, b, c; d; x, y; p, q) = \sum_{r,s=0}^{\infty} \frac{B_{p,q}(a+r+s, d-a)}{B(a, d-a)} (b)_r (c)_s \frac{x^r}{r!} \frac{y^s}{s!}$$

$$(\max\{|x|, |y|\} < 1; \Re(p) \geq 0, \Re(q) \geq 0;)$$

$$(2.4) \quad F_{p,q}(a, b, c, d; e; x, y, z; p, q) = \sum_{r,s,t=0}^{\infty} \frac{B_{p,q}(a+r+s+t, e-a)}{B(a, e-a)} (b)_r (c)_s (d)_t \frac{x^r}{r!} \frac{y^s}{s!} \frac{z^t}{t!}$$

$$(\max\{|x|, |y|, |z|\} < 1; \Re(p) \geq 0, \min\{\Re(\alpha), \Re(\beta), \Re(m) > 0\}.)$$

The product of two Gaussian series¹⁶:

$$(2.5) \quad {}_2F_1(a, b; c; x) {}_2F_1(a', b'; c'; y) = \sum_{m,n=0}^{\infty} \frac{(a)_m (a')_n (b)_m (b')_n}{(c)_m (c')_n} \frac{x^m}{m!} \frac{y^n}{n!}$$

Main Definition

In this section, we have obtained a new extension of the generalized Hurwitz-Lerch Zeta function of two variables (1.3) and three variables (1.5) involving extended Beta function.

Extension of the Hurwitz-Lerch Zeta function of two variables defined in (1.3) is defined as:

$$(2.6) \quad \Phi_{\mu, \eta, \eta', \delta, \delta'; \nu, \xi, \xi'}(z, t, s, a) = \sum_{k,l=0}^{\infty} \frac{(\eta)_k (\eta')_l (\delta)_k (\delta')_l B_{pq}(\mu+k+l, \nu-\mu)}{(\xi)_k (\xi')_l B(\mu, \nu-\mu) (k+l+a)^s} \frac{z^k t^l}{k! l!}$$

$$(p \geq 0, q \geq 0; \mu, \eta, \eta', \delta, \delta' \in \mathbb{C}; \nu, \xi, \xi', a \neq \{0, -1, -2, \dots\}; s \in \mathbb{C} \text{ when } |z, t| < 1);$$

$$\Re(s + \nu + \xi + \xi' - \mu - \eta - \eta' - \delta - \delta') > 1 \text{ when } |z, t| = 1).$$

Extension of the Hurwitz-Lerch Zeta function of three variables given in (1.5) is defined as:

$$(2.7) \quad \Phi_{\mu, \eta, \eta', \eta'', \delta, \delta', \delta''; \nu, \xi, \xi', \xi''}(z, t, r, s, a) \\ = \sum_{k,l,m=0}^{\infty} \frac{(\eta)_k (\eta')_l (\eta'')_m (\delta)_k (\delta')_l (\delta'')_m B_{pq}(\mu+k+l+m, \nu-\mu)}{(\xi)_k (\xi')_l (\xi'')_m B(\mu, \nu-\mu) (k+l+m+a)^s} \frac{z^k t^l r^m}{k! l! m!}$$

$$(p \geq 0, q \geq 0; \mu, \eta, \eta', \eta'', \delta, \delta', \delta'' \in \mathbb{C}; \nu, \xi, \xi', \xi'', a \\ \neq \{0, -1, -2, \dots\}; s \in \mathbb{C} \text{ when } |z, t, r| < 1);$$

$$\Re(s + \nu + \xi + \xi' + \xi'' - \mu - \eta - \eta' - \eta'' - \delta - \delta' - \delta'') > 1 \text{ when } |z, t, r| = 1).$$

Special cases:

- i. In equation (2.6) substituting $\delta = \xi, \delta' = \xi'$, we obtain the extended form of the Generalized Hurwitz-Lerch Zeta function found by Choi and Parmar¹⁴:

$$(2.8) \quad \Phi_{\mu, \eta, \eta'; \nu}(z, t, s, a) = \sum_{k, l=0}^{\infty} \frac{(\eta)_k (\eta')_l B_{pq}(\mu + k + l, \nu - \mu)}{B(\mu, \nu - \mu)(k + l + a)^s} \frac{z^k t^l}{k! l!}$$

$$(p \geq 0, q \geq 0; \mu, \eta, \eta' \in \mathbb{C}; \nu, a \neq \{0, -1, -2, \dots\}; s \in \mathbb{C} \text{ when } |z, t| < 1); \\ \Re(s + \nu - \mu - \eta - \eta') > 1 \text{ when } |z, t| = 1).$$

- ii. In equation (1.7) substituting $\delta = \xi, \delta' = \xi', \delta'' = \xi''$, we obtain a new extended form of the Generalized Hurwitz-Lerch Zeta function equation (1.4):

$$(2.9) \Phi_{\mu, \eta, \eta', \eta''; \nu}(z, t, r, s, a) = \sum_{k, l, m=0}^{\infty} \frac{(\eta)_k (\eta')_l (\eta'')_m B_{pq}(\mu + k + l + m, \nu - \mu)}{B(\mu, \nu - \mu)(k + l + m + a)^s} \frac{z^k t^l r^m}{k! l! m!}$$

$$(p \geq 0, q \geq 0; \mu, \eta, \eta', \eta'' \in \mathbb{C}; \nu, a \neq \{0, -1, -2, \dots\}; s \in \mathbb{C} \text{ when } |z, t, r| < 1); \\ \Re(s + \nu - \mu - \eta - \eta' - \eta'') > 1 \text{ when } |z, t, r| = 1).$$

3. Main Results

3.1 Integral Representation

Here, we have obtained some integral representation of the extension of the Generalized Hurwitz-Lerch zeta function of two variables involving the extended beta function (2.6) as:

Theorem 3.1: For $\Re(p) \geq 0, \Re(q) \geq 0; p = 0, \Re(a) > 0; \Re(s) > 0$, when $|z, t| \leq 1; \Re(s) > 1, z = 1, t = 1$, then

$$(3.1) \quad \Phi_{\mu, \eta, \eta', \delta, \delta'; \nu, \xi, \xi'}(z, t, s, a; p, q) \\ = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} e^{-ax} F_{pq}(\mu, \eta, \eta', \delta, \delta'; \nu, \xi, \xi'; z, t, e^{-ax}) dx$$

Proof: We know that the Eulerian integral of the Gamma function obeys the following identity¹⁷:

$$(3.2) \quad \frac{1}{(k+l+a)^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} e^{-(k+l+a)x} dx \\ (\min\{\Re(s), \Re(a)\} > 0; m \in \mathbb{N}_0)$$

Employing the above result in equation (2.6) and then interchanging the order of summation and integration, we obtain:

$$\Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}[k,l,s,a;p,q] \\ = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-ax} \sum_{k,l=0}^\infty \frac{(\eta)_k(\eta')_l(\delta)_k(\delta')_l B_{pq}(\mu+k+l, \nu-\mu)}{(\xi)_k(\xi')_l B(\mu, \nu-\mu)(k+l+a)^s} \frac{z^k t^l}{k!l!}$$

In view of the definition (2.1) and (2.3), we obtain the required result (3.1),

Theorem 3.2: For $\Re(p) > 0$, $\Re(q) > 0$; $p = 0, q = 0, \Re(\mu) > 0$, $\Re(\nu) > 0$ $\min\{s, a\} > 0$, then

$$(3.3) \quad \Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}(z, t, s, a; p, q) = \frac{e^{-p-q}\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \\ * \int_0^\infty \frac{x^\mu}{(1+x)^\nu} \exp\left(-qx - \frac{p}{x}\right) \Phi_{\eta,\eta',\delta,\delta';\xi,\xi'}\left(\frac{zx}{1+x}, \frac{tx}{1+x}, s, a\right) dx$$

Proof: On putting $\delta_1 = \mu + k + l$ and $\delta_2 = \nu - \mu$ in the underlying integral representation of the extended beta function¹²:

$$(3.4) \quad B_{pq}(\delta_1, \delta_2) = e^{-p-q} \int_0^\infty \frac{x^{\delta_1-1}}{(1+x)^{\delta_1+\delta_2}} \exp\left(-qx - \frac{p}{x}\right) dx$$

we obtain:

$$(3.5) \quad B_{pq}(\mu + k + l, \nu - \mu) = e^{-p-q} \int_0^\infty \frac{x^{\mu+k+l-1}}{(1+x)^{\nu+k+l}} \exp\left(-qx - \frac{p}{x}\right) dx$$

Using (3.5) in (2.6), we obtain:

$$\Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}(z, t, s, a; p, q) = \sum_{k,l=0}^\infty \frac{(\eta)_k(\eta')_l(\delta)_k(\delta')_l}{(\xi)_k(\xi')_l(k+l+a)^s} e^{-p-q} * \\ \int_0^\infty \frac{x^{\mu+k+l-1}}{(1+x)^{\nu+k+l}} \exp\left(-qx - \frac{p}{x}\right) \frac{\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \int_0^\infty \frac{x^{\mu+k+l-1}}{(1+x)^{\nu+k+l}} \frac{z^k t^l}{k!l!} dx$$

$$\Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}(z, t, s, a) = \frac{e^{-p-q}\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \\ * \int_0^\infty \frac{x^\mu}{(1+x)^\nu} \exp\left(-qx - \frac{p}{x}\right) \Phi_{\eta,\eta',\delta,\delta';\xi,\xi'}\left(\frac{zx}{1+x}, \frac{tx}{1+x}, s, a\right) dx$$

which proves Theorem 3.2.

Theorem 3.3: For $\Re(p) > 0, \Re(q) > 0$; $p = 0, q = 0, \Re(\mu) > 0$; $\Re(\nu) > 0$, $\min\{s, a\} > 0$, then

$$(3.6) \quad \Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}(z, t, s, a; p, q) = \frac{e^{-p-q}\Gamma(\nu)}{\Gamma(\mu)\Gamma(s)\Gamma(\nu-\mu)} \int_0^\infty \int_0^\infty \frac{y^{s-1} e^{-ay} x^{\mu-1}}{(1+x)^{\nu-1}} \\ * \exp\left(-qx - \frac{p}{x}\right) \sum_{k=0}^\infty \frac{(\eta)_k(\delta)_k}{(\xi)_k k!} \left(1 - \frac{zxe^{-y}}{1+x}\right)^{-k} \sum_{l=0}^\infty \frac{(\eta')_l(\delta')_l}{(\xi')_l l!} \left(1 - \frac{txe^{-y}}{1+x}\right)^{-l} dx dy$$

Proof: Using (3.5) in (2.6), we obtain:

$$\begin{aligned}
 \Phi_{\mu, \eta, \eta', \delta, \delta'; \nu, \xi, \xi'}(z, t, s, a; p, q) &= \sum_{k, l=0}^{\infty} \frac{(\eta)_k (\eta')_l (\delta)_k (\delta')_l}{(\xi)_k (\xi')_l (k+l+a)^s} e^{-p-q} \\
 &\quad * \int_0^{\infty} \frac{x^{\mu+k+l-1}}{(1+x)^{\nu+k+l}} \exp\left(-qx - \frac{p}{x}\right) \frac{\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \frac{z^k t^l}{k!l!} dx \\
 &= \frac{e^{-p-q}\Gamma(\nu)}{\Gamma(\mu)\Gamma(s)\Gamma(\nu-\mu)} \int_0^{\infty} \int_0^{\infty} \frac{y^{s-1} e^{-ay} x^{\mu-1}}{(1+x)^{\nu-1}} \\
 &\quad * \exp\left(-qx - \frac{p}{x}\right) \sum_{k=0}^{\infty} \frac{(\eta)_k (\delta)_k}{(\xi)_k k!} \left(1 - \frac{zx e^{-y}}{1+x}\right)^{-k} \\
 &\quad * \sum_{l=0}^{\infty} \frac{(\eta')_l (\delta')_l}{(\xi')_l l!} \left(1 - \frac{tx e^{-y}}{1+x}\right)^{-l} dx dy \\
 &\quad \{\text{using (3.2)}\}
 \end{aligned}$$

which proves Theorem 3.3.

Theorem 3.4: For $\Re(p) > 0$, $\Re(q) > 0$; $p = 0$, $q = 0$, $\Re(\mu) > 0$; $\Re(\nu) > 0$, $\min\{s, a\} > 0$, then

$$\begin{aligned}
 (3.7) \Phi_{\mu, \eta, \eta', \delta, \delta'; \nu, \xi, \xi'}(z, t, s, a; p, q) &= \frac{2^{1-\nu}\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \int_{-1}^1 (1+x)^{\mu-1} (1-x)^{\nu-\mu-1} \\
 &\quad * \exp\left(-\frac{2(p+q)+2(q-p)}{(1-x^2)}\right) \Phi_{\eta, \eta', \delta, \delta'; \xi, \xi'}\left(\frac{z(1+x)}{2}, \frac{t(1+x)}{2}, s, a\right) dx
 \end{aligned}$$

Proof: On putting $\delta_1 = \mu + k + l$, $\delta_2 = \nu - \mu$ in the underlying integral representation of the extended Beta function¹²:

$$(3.8) B_{p,q}(\delta_1, \delta_2) = 2^{1-\delta_1-\delta_2} \int_{-1}^1 (1+x)^{\delta_1-1} (1-x)^{\delta_2-1} \exp\left(-\frac{2(p+q)+2(q-p)}{1-x^2}\right) dx$$

Using (3.8) in (2.6), we obtain:

$$\begin{aligned}
 \Phi_{\mu, \eta, \eta', \delta, \delta'; \nu, \xi, \xi'}(z, t, s, a; p, q) &= \sum_{k, l=0}^{\infty} \frac{(\eta)_k (\eta')_l (\delta)_k (\delta')_l}{(\xi)_k (\xi')_l (k+l+a)^s} 2^{1-\nu-k-l} \\
 &\quad * \int_{-1}^1 (1+x)^{\mu+k+l-1} (1-x)^{\nu-\mu-1} \exp\left(-\frac{2(p+q)+2(q-p)}{(1-x^2)}\right) \frac{z^k t^l}{k!l!} dx \\
 \Phi_{\mu, \eta, \eta', \delta, \delta'; \nu, \xi, \xi'}(z, t, s, a; p, q) &= \frac{2^{1-\nu}\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \int_{-1}^1 (1+x)^{\mu-1} (1-x)^{\nu-\mu-1} \\
 &\quad * \exp\left(-\frac{2(p+q)+2(q-p)}{(1-x^2)}\right) \Phi_{\eta, \eta', \delta, \delta'; \xi, \xi'}\left(\frac{z(1+x)}{2}, \frac{t(1+x)}{2}, s, a\right) dx
 \end{aligned}$$

which proves the results (3.7).

Theorem 3.5: For $\Re(p) > 0$, $\Re(q) > 0$; $p = 0$, $q = 0$, $\Re(\mu) > 0$; $\Re(\nu) > 0$, $\min\{s, a\} > 0$, then

$$(3.9) \quad \Phi_{\mu, \eta, \eta', \delta, \delta'; \nu, \xi, \xi'}(z, t, s, a; p, q) = \frac{2^{1-\nu} \Gamma(\nu)}{\Gamma(\mu) \Gamma(s) \Gamma(\nu-\mu)} \\ * \int_{-1}^1 \int_0^\infty y^{s-1} e^{-\nu y} (1+x)^{\mu-1} (1-x)^{\nu-\mu-1} \\ * \exp\left(\frac{-2(p+q)+2(q-p)}{(1-x^2)}\right) \sum_{k=0}^\infty \frac{(\eta)_k (\delta)_k}{(\xi)_k k!} \left(1 - \frac{z(1+x)e^{-y}}{2}\right)^{-k} \\ * \sum_{l=0}^\infty \frac{(\eta')_l (\delta')_l}{(\xi')_l l!} \left(1 - \frac{t(1+x)e^{-y}}{2}\right)^{-l} dx$$

Proof: It can be easy to prove the assertions of Theorem 3.5 and of Theorem 3.4.

Theorem 3.6: For $\Re(p) > 0$, $\Re(q) > 0$; $p = 0$, $q = 0$, $\Re(\mu) > 0$; $\Re(\nu) > 0$, $\min\{s, a\}$, then

$$(3.10) \quad \Phi_{\mu, \eta, \eta', \delta, \delta'; \nu, \xi, \xi'}(z, t, s, a; p, q) = \frac{(c-b)\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \\ * \int_0^c (x-b)^{\mu-1} (c-x)^{\nu-\mu-1} \exp\left(-\frac{(c-b)}{(x-b)(c-x)}\{(q-p)x + (pc-qb)\}\right) \\ * \Phi_{\eta, \eta', \delta, \delta'; \xi, \xi'}\left(z\left(\frac{x-b}{c-a}\right), t\left(\frac{x-b}{c-a}\right), s, a\right) dx$$

Proof: On putting $\delta_1 = \mu + k + l$, $\delta_2 = \nu - \mu$ in the underlying integral representation of the extended Beta function¹²:

$$(3.11) \quad B_{p,q}(\delta_1, \delta_2) = (c-b)^{1-\delta_1-\delta_2} \int_a^c (x-b)^{\delta_1-1} (c-x)^{\delta_2-1} \\ * \exp\left(-\frac{(c-b)}{(x-b)(c-x)}\{(q-p)x + (pc-qb)\}\right) dx$$

Using (3.11) in (2.6), we obtain:

$$\Phi_{\mu, \eta, \eta', \delta, \delta'; \nu, \xi, \xi'}(z, t, s, a; p, q) \\ = \sum_{k,l=0}^\infty \frac{(\eta)_k (\eta')_l (\delta)_k (\delta')_l B_{pq}(\mu+k+l, \nu-\mu)}{(\xi)_k (\xi')_l B(\mu, \nu-\mu)(k+l+a)^s} \frac{z^k t^l}{k! l!} \\ = \sum_{k,l=0}^\infty \frac{(\eta)_k (\eta')_l (\delta)_k (\delta')_l}{(\xi)_k (\xi')_l (k+l+a)^s} (c-b)^{1-k-l-\nu} \\ * \int_a^c (x-b)^{\mu+k+l-1} (c-x)^{\nu-\mu-1} \exp\left(-\frac{(c-b)}{(x-b)(c-x)}\{(q-p)x + (pc-qb)\}\right) \frac{z^k t^l}{k! l!} dx$$

$$\begin{aligned} \Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}(z,t,s,a) &= \frac{(c-b)^{1-\nu}\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \\ &* \int_0^c (x-b)^{\mu-1}(c-x)^{\nu-\mu-1} \exp\left(-\frac{(c-b)}{(x-b)(c-x)}\{(q-p)x+(pc-qb)\}\right) \\ &* \Phi_{\eta,\eta',\delta,\delta';\xi,\xi'}\left(z\left(\frac{x-b}{c-b}\right), t\left(\frac{x-b}{c-b}\right), s, a\right) dx, \quad \{\text{using (2.6)}\} \end{aligned}$$

which is the desired results (3.10), i.e. Theorem 3.6.

Theorem 3.7: For $\Re(p) > 0$, $\Re(q) > 0$; $p = 0$, $q = 0$, $\Re(\mu) > 0$; $\Re(\nu) > 0$, $\min\{s, a\} > 0$, then

$$\begin{aligned} (3.12) \quad \Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi'}(z,t,s,a;p,q) &= \frac{(c-a)\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \\ &* \int_0^c (x-a)^{\mu-1}(c-x)^{\nu-\mu-1} \exp\left(-\frac{(c-a)}{(x-a)(c-x)}\{(q-p)x+(pc-qa)\}\right) \\ &* \sum_{k=0}^{\infty} \frac{(\eta)_k(\delta)_k}{(\xi)_k k!} \left(1 - z\left(\frac{x-a}{c-a}\right)e^{-y}\right)^{-k} \sum_{l=0}^{\infty} \frac{(\eta')_l(\delta')_l}{(\xi')_l l!} \left(1 - t\left(\frac{x-a}{c-a}\right)e^{-y}\right)^{-l} dx \end{aligned}$$

Proof: It is easy to prove the assertions of Theorem 3.7 and of Theorem 3.6.

3.2 Derivative formula:

Theorem 3.8: The Following differential formula for $\Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi';p,q}(z,t,s,a;p,q)$ holds:

$$\begin{aligned} (3.13) \quad \frac{d^{k+l}}{dz^k dt^l} \left\{ \Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi';p,q}(z,t,s,a;p,q) \right\} &= \frac{(\mu)_{k+l}(\eta)_k(\eta')_l(\delta)_k(\delta')_l}{(v)_{k+l}(\xi)_k(\xi')_l} \\ &* \Phi_{\mu+k+l,\delta+k,\delta'+l;\nu+k+l,\xi+k,\xi'+l;p,q}(z,t,s,a+k+l;p,q) \quad (m \in \mathbb{N}) \end{aligned}$$

Proof: Consider the derivative of (2.6) with respect to z, t , we obtain:

$$\begin{aligned} \frac{d^{k+l}}{dz^k dt^l} \left\{ \Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi';p,q}(z,t,s,a;p,q) \right\} &= \sum_{k,l=0}^{\infty} \frac{(\eta)_k(\eta')_l(\delta)_k(\delta')_l B_{pq}(\mu+k+l, \nu-\mu)}{(\xi)_k(\xi')_l B(\mu, \nu-\mu)(k+l+a)^s} \\ &* \frac{z^{k-1} t^{l-1}}{(k-1)!(l-1)!} \end{aligned}$$

which upon replacing k by $k+1$ and l by $l+1$ in the above equation and employing the identity

$$B(\mu, \nu - \mu) = \frac{\nu}{\mu} B(\mu + 1, \nu - \mu), \quad \nu_{m+1} = \nu(\nu + 1)_m,$$

Leads to the derivative formula

$$\frac{d}{dz^k dt^l} \left\{ \Phi_{\mu,\eta,\eta',\delta,\delta';\nu,\xi,\xi';p,q}(z,t,s,a;p,q) \right\} = \frac{(\eta)_k(\eta')_l(\delta)_k(\delta')_l}{(\xi)_k(\xi')_l}$$

$$* \Phi_{\mu+1, \eta+1, \eta'+1, \delta+1, \delta'+1; \nu+1, \xi+1, \xi'+1; p, q}(z, t, s, a+1; p, q)$$

Recursive application of this procedure provides the required results (3.13).

3.3. Mellin transform of the Hurwitz-Lerch Zeta Function:

Theorem 3.9: For $\Re(p) > 0, \Re(q) > 0, \Re(\varphi) > 0, \Re(\psi) > 0, \Re(\mu + k + l + \varphi) > 0, \Re(\nu + k + l + \varphi) > 0$, then the Mellin transform of the function $\Phi_{\mu, \eta, \eta', \delta, \delta'; \nu, \xi, \xi'; p, q}(z, t, s, a; p, q)$ is given by

$$(3.14) \quad M\left\{\Phi_{\mu, \eta, \eta', \delta, \delta'; \nu, \xi, \xi'}(z, t, s, a; p, q): p \rightarrow \varphi, q \rightarrow \psi\right\} \\ = \frac{\Gamma(\varphi)\Gamma(\psi)B(\mu + \varphi, \nu - \mu + \psi)}{B(\mu, \nu - \mu)} \Phi_{\mu + \varphi, \eta, \eta', \delta, \delta'; \nu + \varphi + \psi, \xi, \xi'}(z, t, s, a).$$

Proof: The Mellin transform of a suitable integrable function $f(k)$ with index φ is defined, generally, by

$$(3.15) \quad M\{f(k): k \rightarrow \varphi\} = \int_0^\infty k^{\varphi-1} f(k) dk \quad (0 < k < \infty).$$

Using (2.6) in (3.15), we obtain:

$$\begin{aligned} & M\left\{\Phi_{\mu, \eta, \eta', \delta, \delta'; \nu, \xi, \xi'}(z, t, s, a; p, q): p \rightarrow \varphi, q \rightarrow \psi\right\} \\ &= \int_0^\infty \int_0^\infty p^{\varphi-1} q^{\psi-1} \left\{\Phi_{\mu, \eta, \eta', \delta, \delta'; \nu, \xi, \xi'}(z, t, s, a; p, q): p, q\right\} dp dq \\ &= \int_0^\infty \int_0^\infty p^{\varphi-1} q^{\psi-1} \sum_{k, l=0}^\infty \frac{(\eta)_k (\eta')_l (\delta)_k (\delta')_l B_{p, q}(\mu + k + l, \nu - \mu)}{(\xi)_k (\xi')_l B(\mu, \nu - \mu) (k + l + a)^s} \frac{z^k t^l}{k! l!} dp dq \\ &= \sum_{k, l=0}^\infty \frac{(\eta)_k (\eta')_l (\delta)_k (\delta')_l}{(\xi)_k (\xi')_l B(\mu, \nu - \mu) (k + l + a)^s} \frac{z^k t^l}{k! l!} \\ &\quad * \int_0^\infty \int_0^\infty p^{\varphi-1} q^{\psi-1} B_{p, q}(\mu + k + l, \nu - \mu) dp dq \end{aligned}$$

Employing the underlying well-known integral representations¹²:

$$\begin{aligned} & \int_0^\infty \int_0^\infty p^{\varphi-1} q^{\psi-1} B_{p, q}(\mu + k + l, \nu - \mu) dp dq \\ &= \Gamma(\varphi)\Gamma(\psi)B(\mu + k + l + \varphi, \nu - \mu + \psi)(\Re(\varphi) > 0, \Re(\psi) > 0) \end{aligned}$$

we obtain:

$$\begin{aligned} & M\left\{\Phi_{\mu, \eta, \eta', \delta, \delta'; \nu, \xi, \xi'}(z, t, s, a; p, q): p \rightarrow \varphi, q \rightarrow \psi\right\} \\ &= \frac{\Gamma(\varphi)\Gamma(\psi)}{B(\mu, \nu - \mu)} \sum_{k, l=0}^\infty \frac{(\eta)_k (\eta')_l (\delta)_k (\delta')_l B_{p, q}(\mu + k + l + \varphi, \nu - \mu + \psi)}{(\xi)_k (\xi')_l (k + l + a)^s} \frac{z^k t^l}{k! l!} dp dq \end{aligned}$$

$$\begin{aligned}
&= \frac{\Gamma(\varphi)\Gamma(\psi)B(\mu+\varphi, \nu-\mu+\psi)}{B(\mu, \nu-\mu)} \sum_{k,l=0}^{\infty} \frac{(\eta)_k(\eta')_l(\delta)_k(\delta')_l}{(\xi)_k(\xi')_l(\varphi+\nu+\psi)_k(\varphi+\nu+\psi)_l(k+l+a)^s} \frac{(\mu+\varphi)_{k+l}}{k!l!} \frac{z^k t^l}{k!l!} \\
M \left\{ \Phi_{\mu, \eta, \eta', \delta, \delta'; \nu, \xi, \xi'}(z, t, s, a; p, q) : p \rightarrow \varphi, q \rightarrow \psi \right\} &= \frac{\Gamma(\varphi)\Gamma(\psi)B(\mu+\varphi, \nu-\mu+\psi)}{B(\mu, \nu-\mu)} \\
&\quad * \Phi_{\mu+\varphi, \eta, \eta', \delta, \delta'; \nu+\varphi+\psi, \xi, \xi'}(z, t, s, a)
\end{aligned}$$

which proves Theorem 3.9.

3.4 Integral representations:

The section deals with the integral representation of the new extension of the generalized Hurwitz-Lerch zeta function of three variables involving the extended beta function (2.7) as follows:

Theorem 3.10. For $\Re(p) \geq 0, \Re(q) \geq 0; p = 0, \Re(a) > 0; \Re(s) > 0$, when $|z, t, r| \leq 1; \Re(s) > 1$, when $z = 1, t = 1, r = 1$ then

$$\begin{aligned}
(3.16) \quad &\Phi_{\mu, \eta, \eta', \eta'', \delta, \delta', \delta''; \nu, \xi, \xi', \xi''}(k, l, m, s, a; p, q) \\
&= \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} e^{-ax} F_{pq}(\mu, \eta, \eta', \eta'', \delta, \delta', \delta''; \nu, \xi, \xi', \xi''; z, t, r, e^{-ax}) dx
\end{aligned}$$

Proof: We know that Eulerian integral of the Gamma function obeys the following identity¹⁸;

$$\begin{aligned}
(3.17) \quad &\frac{1}{(k+l+m+a)^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} e^{-(k+l+m+a)x} dx \\
&\quad (\min\{\Re(s), \Re(a)\} > 0; m \in \mathbb{N}_0)
\end{aligned}$$

Employing the above result in equation (2.7) and then interchanging the order of summation and integration, we obtain:

$$\begin{aligned}
(3.18) \quad &\Phi_{\mu, \eta, \eta', \eta'', \delta, \delta', \delta''; \nu, \xi, \xi', \xi''}(k, l, m, s, a; p, q) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} e^{-ax} \\
&\quad * \sum_{k,l,m=0}^{\infty} \frac{(\eta)_k(\eta')_l(\eta'')_m(\delta)_k(\delta')_l(\delta'')_m B_{pq}(\mu+k+l+m, \nu-\mu)}{(\xi)_k(\xi')_l(\xi'')_m B(\mu, \nu-\mu)(k+l+m+a)^s} \frac{z^k t^l r^m}{k!l!m!}
\end{aligned}$$

In view of the definition (2.1) and (2.4), we obtain the required result (3.16).

Theorem 3.11: For $\Re(p) > 0, \Re(q) > 0; p = 0, q = 0, \Re(\mu) > 0; \Re(\nu) > 0, \min\{s, a\} > 0$, then

$$\begin{aligned}
(3.19) \quad &\Phi_{\mu, \eta, \eta', \eta'', \delta, \delta', \delta''; \nu, \xi, \xi', \xi''}(z, t, r, s, a; p, q) = \frac{e^{-p-q}\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \\
&\quad * \int_0^{\infty} \frac{x^{\mu}}{(1+x)^{\nu}} \exp\left(-qx - \frac{p}{x}\right) \Phi_{\eta, \eta', \eta'', \delta, \delta', \delta''; \xi, \xi', \xi''}\left(\frac{zx}{1+x}, \frac{tx}{1+x}, \frac{rx}{1+x}, s, a\right) dx
\end{aligned}$$

Proof: On putting $\delta_1 = \mu + k + l + m$ and $\delta_2 = \nu - \mu$ in (3.4), we obtain:

$$(3.20) \quad B_{pq}(\mu + k + l + m, \nu - \mu) = e^{-p-q} \int_0^\infty \frac{x^{\mu+k+l+m-1}}{(1+x)^{\nu+k+l+m}} \exp\left(-qx - \frac{p}{x}\right) dx$$

using (3.20) in (2.7), we obtain:

$$(3.21) \quad \Phi_{\mu, \eta, \eta', \eta'', \delta, \delta', \delta'', \nu, \xi, \xi', \xi''}(z, t, r, s, a; p, q) \\ = \sum_{k, l, m=0}^\infty \frac{(\eta)_k (\eta')_l (\eta'')_m (\delta)_k (\delta')_l (\delta'')_m}{(\xi)_k (\xi')_l (\xi'')_m (k+l+m+a)^s} \\ * e^{-p-q} \int_0^\infty \frac{x^{\mu+k+l+m-1}}{(1+x)^{\nu+k+l+m}} \exp\left(-qx - \frac{p}{x}\right) \\ * \frac{\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \int_0^\infty \frac{x^{\mu+k+l+m-1}}{(1+x)^{\nu+k+l+m}} \frac{z^k t^l r^m}{k! l! m!} dx \\ \Phi_{\mu, \eta, \eta', \eta'', \delta, \delta', \delta'', \nu, \xi, \xi', \xi''}(z, t, r, s, a; p, q) = \frac{e^{-p-q} \Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \\ * \int_0^\infty \frac{x^\mu}{(1+x)^\nu} \exp\left(-qx - \frac{p}{x}\right) \Phi_{\eta, \eta', \eta'', \delta, \delta', \delta'', \xi, \xi', \xi''}\left(\frac{zx}{1+x}, \frac{tx}{1+x}, \frac{rx}{1+x}, s, a\right) dx$$

which proves Theorem 3.11.

Theorem 3.12: For $\Re(p) > 0, \Re(q) > 0; p = 0, q = 0, \Re(\mu) > 0; \Re(\nu) > 0, \min\{s, a\} > 0$, then

$$(3.22) \quad \Phi_{\mu, \eta, \eta', \eta'', \delta, \delta', \delta'', \nu, \xi, \xi', \xi''}(z, t, r, s, a; p, q) \\ = \frac{e^{-p-q} \Gamma(\nu)}{\Gamma(\mu)\Gamma(s)\Gamma(\nu-\mu)} \int_0^\infty \int_0^\infty \frac{y^{s-1} e^{-ay} x^{\mu-1}}{(1+x)^{\nu-1}} \\ * \exp\left(-qx - \frac{p}{x}\right) \sum_{k=0}^\infty \frac{(\eta)_k (\delta)_k}{(\xi)_k k!} \left(1 - \frac{zxe^{-y}}{1+x}\right)^{-k} \sum_{l=0}^\infty \frac{(\eta')_l (\delta')_l}{(\xi')_l l!} \left(1 - \frac{txe^{-y}}{1+x}\right)^{-l} \\ * \sum_{m=0}^\infty \frac{(\eta'')_m (\delta'')_m}{(\xi'')_m m!} \left(1 - \frac{mxe^{-y}}{1+x}\right)^{-m} dx dy$$

Proof: Using (3.21) in (2.7), we obtain:

$$\Phi_{\mu, \eta, \eta', \eta'', \delta, \delta', \delta'', \nu, \xi, \xi', \xi''}(z, t, r, s, a; p, q) \\ = \sum_{k, l, m=0}^\infty \frac{(\eta)_k (\eta')_l (\eta'')_m (\delta)_k (\delta')_l (\delta'')_m}{(\xi)_k (\xi')_l (\xi'')_m (k+l+m+a)^s} e^{-p-q} \int_0^\infty \frac{x^{\mu+k+l+m-1}}{(1+x)^{\nu+k+l+m}} \exp\left(-qx - \frac{p}{x}\right) \\ * \frac{\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \frac{z^k t^l r^m}{k! l! m!} dx \\ \Phi_{\mu, \eta, \eta', \eta'', \delta, \delta', \delta'', \nu, \xi, \xi', \xi''}(z, t, r, s, a; p, q) \\ = \frac{e^{-p-q} \Gamma(\nu)}{\Gamma(\mu)\Gamma(s)\Gamma(\nu-\mu)} \int_0^\infty \int_0^\infty \frac{y^{s-1} e^{-ay} x^{\mu-1}}{(1+x)^{\nu-1}}$$

$$\begin{aligned}
& * \exp\left(-qx - \frac{p}{x}\right) \sum_{k=0}^{\infty} \frac{(\eta)_k (\delta)_k}{(\xi)_k k!} \left(1 - \frac{zxe^{-y}}{1+x}\right)^{-k} \\
& * \sum_{l=0}^{\infty} \frac{(\eta')_l (\delta')_l}{(\xi')_l l!} \left(1 - \frac{txe^{-y}}{1+x}\right)^{-l} \\
& * \sum_{m=0}^{\infty} \frac{(\eta'')_m (\delta'')_m}{(\xi'')_m m!} \left(1 - \frac{mxe^{-y}}{1+x}\right)^{-m} dx dy, \quad \{\text{using (3.17)}\}
\end{aligned}$$

which proves Theorem 3.12.

Theorem 3.13: For $\Re(p) > 0, \Re(q) > 0; p = 0, q = 0, \Re(\mu) > 0; \Re(\nu) > 0, \min\{s, a\} > 0$, then

$$\begin{aligned}
(3.23) \quad & \Phi_{\mu, \eta, \eta', \eta'', \delta, \delta', \delta'', \nu, \xi, \xi', \xi''}(z, t, r, s, a; p, q) \\
& = \frac{2^{1-\nu} \Gamma(\nu)}{\Gamma(\mu) \Gamma(\nu-\mu)} \int_{-1}^1 (1+x)^{\mu-1} (1-x)^{\nu-\mu-1} \exp\left(-\frac{2(p+q)+2(q-p)}{(1-x^2)}\right) \\
& \quad * \Phi_{\eta, \eta', \eta'', \delta, \delta', \delta'', \xi, \xi', \xi''}\left(\frac{z(1+x)}{2}, \frac{t(1+x)}{2}, \frac{r(1+x)}{2}, s, a\right) dx
\end{aligned}$$

Proof: On putting $\delta_1 = \mu + k + l + m$, $\delta_2 = \nu - \mu$ in (3.8), we obtain:

$$\begin{aligned}
(3.24) \quad & B_{p,q}(\mu + k + l + m, \nu - \mu) \\
& = 2^{1-k-l-m-\nu} \int_{-1}^1 (1+x)^{\mu+k+l+m-1} (1-x)^{\nu-\mu-1} \\
& \quad * \exp\left(-\frac{2(p+q)+2(q-p)}{1-x^2}\right) dx
\end{aligned}$$

Using (3.24) in (2.7), we obtain:

$$\begin{aligned}
& \Phi_{\mu, \eta, \eta', \eta'', \delta, \delta', \delta'', \nu, \xi, \xi', \xi''}(z, t, r, s, a; p, q) \\
& = \sum_{k,l,m=0}^{\infty} \frac{(\eta)_k (\eta')_l (\eta'')_m (\delta)_k (\delta')_l (\delta'')_m}{(\xi)_k (\xi')_l (\xi'')_m (k+l+m+a)^s} \\
& \quad * 2^{1-\nu-k-l} \int_{-1}^1 (1+x)^{\mu+k+l+m-1} (1-x)^{\nu-\mu-1} \\
& \quad * \exp\left(-\frac{2(p+q)+2(q-p)}{(1-x^2)}\right) \frac{z^k t^l r^m}{k! l! m!} dx \\
& \Phi_{\mu, \eta, \eta', \eta'', \delta, \delta', \delta'', \nu, \xi, \xi', \xi''}(z, t, r, s, a; p, q) \\
& = \frac{2^{1-\nu} \Gamma(\nu)}{\Gamma(\mu) \Gamma(\nu-\mu)} \int_{-1}^1 (1+x)^{\mu-1} (1-x)^{\nu-\mu-1} \\
& \quad * \exp\left(-\frac{2(p+q)+2(q-p)}{(1-x^2)}\right) \Phi_{\eta, \eta', \eta'', \delta, \delta', \delta'', \xi, \xi', \xi''}\left(\frac{z(1+x)}{2}, \frac{t(1+x)}{2}, \frac{r(1+x)}{2}, s, a\right) dx
\end{aligned}$$

which proves the results (3.23).

Theorem 3.14: For $\Re(p) > 0, \Re(q) > 0; p = 0, q = 0, \Re(\mu) > 0; \Re(v) > 0, \min\{s, a\} > 0$, then

$$(3.25) \quad \Phi_{\mu, \eta, \eta', \eta'', \delta, \delta', \delta'', \nu, \xi, \xi', \xi''}(z, t, r, s, a; p, q) \\ = \frac{2^{1-\nu} \Gamma(\nu)}{\Gamma(\mu) \Gamma(s) \Gamma(\nu-\mu)} \int_{-1}^1 \int_0^\infty y^{s-1} e^{-\nu y} (1+x)^{\mu-1} (1-x)^{\nu-\mu-1} \\ * \exp\left(\frac{-2(p+q)+2(q-p)}{(1-x^2)}\right) \sum_{k=0}^\infty \frac{(\eta)_k (\delta)_k}{(\xi)_k k!} \left(1 - \frac{z(1+x)e^{-y}}{2}\right)^{-k} \\ * \sum_{l=0}^\infty \frac{(\eta')_l (\delta')_l}{(\xi')_l l!} \left(1 - \frac{t(1+x)e^{-y}}{2}\right)^{-l} \sum_{m=0}^\infty \frac{(\eta'')_m (\delta'')_m}{(\xi'')_m m!} \left(1 - \frac{r(1+x)e^{-y}}{2}\right)^{-m} dx$$

Proof: It can be easy to prove the assertions of Theorem 3.14 and of Theorem 3.13.

Theorem 3.15: For $\Re(p) > 0, \Re(q) > 0; p = 0, q = 0, \Re(\mu) > 0; \Re(v) > 0, \min\{s, a\} > 0$, then

$$(3.26) \quad \Phi_{\mu, \eta, \eta', \eta'', \delta, \delta', \delta'', \nu, \xi, \xi', \xi''}(z, t, r, s, a; p, q) = \frac{(c-b)\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \\ * \int_0^c (x-b)^{\mu-1} (c-x)^{\nu-\mu-1} \exp\left(-\frac{(c-b)}{(x-b)(c-x)}\{(q-p)x + (pc-qb)\}\right) \\ * \Phi_{\eta, \eta', \eta'', \delta, \delta', \delta'', \xi, \xi', \xi''}\left(z\left(\frac{x-b}{c-a}\right), t\left(\frac{x-b}{c-a}\right), r\left(\frac{x-b}{c-a}\right), s, a\right) dx$$

Proof: On putting $\delta_1 = \mu + k + l + m$, $\delta_2 = \nu - \mu$ in (3.11), we obtain:

$$(3.27) \quad B_{p,q}(\mu + k + l + m, \nu - \mu) \\ = (c-b)^{1-k-l-m-\nu} \int_a^c (x-b)^{\mu+k+l+m-1} (c-x)^{\nu-\mu-1} \\ * \exp\left(-\frac{(c-b)}{(x-b)(c-x)}\{(q-p)x + (pc-qb)\}\right) dx$$

Using (3.27) in (2.7), we obtain:

$$\Phi_{\mu, \eta, \eta', \eta'', \delta, \delta', \delta'', \nu, \xi, \xi', \xi''}(z, t, r, s, a; p, q) \\ = \sum_{k,l,m=0}^\infty \frac{(\eta)_k (\eta')_l (\eta'')_m (\delta)_k (\delta')_l (\delta'')_m}{(\xi)_k (\xi')_l (\xi'')_m (k+l+m+a)^s} \\ * (c-b)^{1-k-l-m-\nu} \int_a^c (x-b)^{\mu+k+l+m-1} (c-x)^{\nu-\mu-1} \\ * \exp\left(-\frac{(c-b)}{(x-b)(c-x)}\{(q-p)x + (pc-qb)\}\right) \frac{z^k t^l r^m}{k! l! m!} dx \\ \Phi_{\mu, \eta, \eta', \eta'', \delta, \delta', \delta'', \nu, \xi, \xi', \xi''}(z, t, r, s, a; p, q) = \frac{(c-b)^{1-\nu} \Gamma(\nu)}{\Gamma(\mu) \Gamma(\nu-\mu)}$$

$$\begin{aligned}
& * \int_0^c (x-b)^{\mu-1} (c-x)^{\nu-\mu-1} \exp\left(-\frac{(c-b)}{(x-b)(c-x)}\{(q-p)x + (pc-qb)\}\right) \\
& * \Phi_{\mu,\eta',\eta'',\delta,\delta',\delta'',\xi,\xi',\xi''}\left(z\left(\frac{x-b}{c-b}\right), t\left(\frac{x-b}{c-b}\right), r\left(\frac{x-b}{c-b}\right), s, a\right) dx \\
& \hspace{15em} \{\text{using (2.7)}\}
\end{aligned}$$

which is the desired results (3.26), i.e. Theorem 3.15.

Theorem 3.16: For $\Re(p) > 0, \Re(q) > 0; p = 0, q = 0, \Re(\mu) > 0; \Re(\nu) > 0, \min\{s, a\} > 0$, then

$$\begin{aligned}
(3.28) \quad & \Phi_{\mu,\eta',\eta'',\delta,\delta',\delta'',\nu,\xi,\xi',\xi''}(z, t, r, s, a; p, q) = \frac{(c-a)\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \\
& * \int_0^c (x-a)^{\mu-1} (c-x)^{\nu-\mu-1} \exp\left(-\frac{(c-a)}{(x-a)(c-x)}\{(q-p)x + (pc-qa)\}\right) \\
& * \sum_{k=0}^{\infty} \frac{(\eta)_k(\delta)_k}{(\xi)_k k!} \left(1 - z\left(\frac{x-a}{c-a}\right)e^{-y}\right)^{-k} \sum_{l=0}^{\infty} \frac{(\eta')_l(\delta')_l}{(\xi')_l l!} \left(1 - t\left(\frac{x-a}{c-a}\right)e^{-y}\right)^{-l} \\
& * \sum_{m=0}^{\infty} \frac{(\eta'')_m(\delta'')_m}{(\xi'')_m m!} \left(1 - r\left(\frac{x-a}{c-a}\right)e^{-y}\right)^{-m} dx
\end{aligned}$$

Proof: It is easy to prove the assertions of Theorem 3.16 and of Theorem 3.15.

3.5 Derivative formula:

Theorem 3.17: The Following differential formula for $\Phi_{\mu,\eta',\eta'',\delta,\delta',\delta'',\nu,\xi,\xi',\xi'';p,q}(z, t, r, s, a; p, q)$ holds:

$$\begin{aligned}
(3.29) \quad & \frac{d^{k+l+m}}{dz^k dt^l dr^m} \left\{ \Phi_{\mu,\eta',\eta'',\delta,\delta',\delta'',\nu,\xi,\xi',\xi'';p,q}(k, l, m, s, a; p, q) \right\} \\
& = \frac{(\mu)_{k+l+m}(\eta)_k(\eta')_m(\delta)_k(\delta')_l(\delta'')_m}{(v)_{k+l}(\xi)_k(\xi')_l(\xi'')_m}
\end{aligned}$$

$$* \Phi_{\mu+k+l+m,\delta+k,\delta'+l,\delta''+m;\nu+k+l+m,\xi+k,\xi'+l,\xi''+m;p,q}(z, t, r, s, a + k + l + m; p, q)$$

($m \in \mathbb{N}$)

Proof: Consider the derivative of (2.7) with respect to z, t, r , we obtain:

$$\begin{aligned}
& \frac{d^{k+l+m}}{dz^k dt^l dr^m} \left\{ \Phi_{\mu,\eta',\eta'',\delta,\delta',\delta'',\nu,\xi,\xi',\xi'';p,q}(k, l, m, s, a; p, q) \right\} \\
& = \sum_{k,l,m=0}^{\infty} \frac{(\eta)_k(\eta')_l(\eta'')_m(\delta)_k(\delta')_l(\delta'')_m B_{pq}(\mu+k+l+m, \nu-\mu)}{(\xi)_k(\xi')_l(\xi'')_m B(\mu, \nu-\mu)(k+l+m+a)^s} \\
& * \frac{z^{k-1} t^{l-1} r^{m-1}}{(k-1)!(l-1)!(m-1)!}
\end{aligned}$$

which upon replacing k by $k+1$ and l by $l+1$ in above equation and

employing the identity

$$(3.30) \quad B(\mu, \nu - \mu) = \frac{\nu}{\mu} B(\mu + 1, \nu - \mu), \quad \nu_{m+1} = \nu(\nu + 1)_m,$$

leads to the derivative formula

$$\begin{aligned} & \frac{d}{dz} \frac{d}{dt} \frac{d}{dr} \left\{ \Phi_{\mu, \eta, \eta', \eta'', \delta, \delta', \delta'', \nu, \xi, \xi', \xi'', p, q}(k, l, m, s, a; p, q) \right\} \\ &= \frac{(\eta)_k (\eta')_l (\eta'')_m (\delta)_k (\delta')_l (\delta'')_m}{(\xi)_k (\xi')_l (\xi'')_m} \\ & \quad * \Phi_{\mu+1, \eta+1, \eta'+1, \eta''+1, \delta+1, \delta'+1, \delta''+1, \nu+1, \xi+1, \xi'+1, \xi''+1; p, q}(z, t, r, s, a + 1; p, q) \end{aligned}$$

Recursive application of this procedure provides the required results (3.29).

3.6 Mellin transform of the Hurwitz-Lerch Zeta Function:

Theorem 3.18: For $\Re(p) > 0, \Re(q) > 0, \Re(\varphi) > 0, \Re(\psi) > 0, \Re(\mu + k + l + m + \varphi) > 0, \Re(\nu + k + l + m + \varphi) > 0$, then the Mellin transform of the function $\Phi_{\mu, \eta, \eta', \eta'', \delta, \delta', \delta'', \nu, \xi, \xi', \xi'', p, q}(k, l, m, s, a; p, q)$ is given by

$$(3.31) \quad M \left\{ \Phi_{\mu, \eta, \eta', \eta'', \delta, \delta', \delta'', \nu, \xi, \xi', \xi'', p, q}(z, t, r, s, a; p, q): p \rightarrow \varphi, q \rightarrow \psi \right\} \\ = \frac{\Gamma(\varphi) \Gamma(\psi) B(\mu + \varphi, \nu - \mu + \psi)}{B(\mu, \nu - \mu)} \Phi_{\mu + \varphi, \eta, \eta', \eta'', \delta, \delta', \delta'', \nu + \varphi + \psi, \xi, \xi', \xi'', p, q}(z, t, r, s, a)$$

Proof: Using (2.7) in (3.15), we obtain:

$$\begin{aligned} & M \left\{ \Phi_{\mu, \eta, \eta', \eta'', \delta, \delta', \delta'', \nu, \xi, \xi', \xi'', p, q}(z, t, r, s, a; p, q): p \rightarrow \varphi, q \rightarrow \psi \right\} \\ &= \int_0^\infty \int_0^\infty p^{\varphi-1} q^{\psi-1} \left\{ \Phi_{\mu, \eta, \eta', \eta'', \delta, \delta', \delta'', \nu, \xi, \xi', \xi'', p, q}(z, t, r, s, a; p, q): p, q \right\} dp dq \\ &= \int_0^\infty \int_0^\infty p^{\varphi-1} q^{\psi-1} \sum_{k, l, m=0}^\infty \frac{(\eta)_k (\eta')_l (\eta'')_m (\delta)_k (\delta')_l (\delta'')_m B_{p, q}(\mu + k + l + m, \nu - \mu)}{(\xi)_k (\xi')_l (\xi'')_m B(\mu, \nu - \mu) (k + l + m + a)^s} \\ & \quad * \frac{z^k t^l r^m}{k! l! m!} dp dq \\ &= \sum_{k, l, m=0}^\infty \frac{(\eta)_k (\eta')_l (\eta'')_m (\delta)_k (\delta')_l (\delta'')_m}{(\xi)_k (\xi')_l (\xi'')_m B(\mu, \nu - \mu) (k + l + m + a)^s} \frac{z^k t^l}{k! l!} \\ & \quad * \int_0^\infty \int_0^\infty p^{\varphi-1} q^{\psi-1} B_{p, q}(\mu + k + l + m, \nu - \mu) dp dq \end{aligned}$$

Employing the underlying well-known integral representations¹²:

$$\begin{aligned} & \int_0^\infty \int_0^\infty p^{\varphi-1} q^{\psi-1} B_{p, q}(\mu + k + l + m, \nu - \mu) dp dq \\ &= \Gamma(\varphi) \Gamma(\psi) B(\mu + k + l + m + \varphi, \nu - \mu + \psi) (\Re(\varphi) > 0, \Re(\psi) > 0) \end{aligned}$$

we obtain:

$$\begin{aligned}
 M \left\{ \Phi_{\mu, \eta, \eta', \eta'', \delta, \delta', \delta''; \nu, \xi, \xi', \xi''}(z, t, r, s, a; p, q): p \rightarrow \varphi, q \rightarrow \psi \right\} & \frac{z^k t^l r^m}{k! l! m!} dp dq \\
 &= \frac{\Gamma(\varphi) \Gamma(\psi)}{B(\mu, \nu - \mu)} \sum_{k, l, m=0}^{\infty} \frac{(\eta)_k (\eta')_l (\eta'')_m (\delta)_k (\delta')_l (\delta'')_m B_{p, q}(\mu + k + l + \varphi, \nu - \mu + \psi)}{(\xi)_k (\xi')_l (\xi'')_m (k + l + m + a)^s} \\
 &= \frac{\Gamma(\varphi) \Gamma(\psi) B(\mu + \varphi, \nu - \mu + \psi)}{B(\mu, \nu - \mu)} \\
 &\quad * \sum_{k, l, m=0}^{\infty} \frac{(\eta)_k (\eta')_l (\eta'')_m (\delta)_k (\delta')_l (\delta'')_m (\mu + \varphi)_{k+l+m}}{(\xi)_k (\xi')_l (\xi'')_m (\varphi + \nu + \psi)_k (\varphi + \nu + \psi)_l (k + l + m + a)^s} \frac{z^k t^l r^m}{k! l! m!} \\
 M \left\{ \Phi_{\mu, \eta, \eta', \eta'', \delta, \delta', \delta''; \nu, \xi, \xi', \xi''}(z, t, r, s, a; p, q): p \rightarrow \varphi, q \rightarrow \psi \right\} \\
 &= \frac{\Gamma(\varphi) \Gamma(\psi) B(\mu + \varphi, \nu - \mu + \psi)}{B(\mu, \nu - \mu)} \Phi_{\mu + \varphi, \eta, \eta', \eta'', \delta, \delta', \delta''; \nu + \varphi + \psi, \xi, \xi', \xi''}(z, t, r, s, a)
 \end{aligned}$$

which proves Theorem 3.18.

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