

Certain Expansions of Sylvester Polynomials in terms of Cesaro Polynomials, Bernoulli Polynomials, Bateman's Polynomials and Generalized Bessel Polynomials

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Abstract: In the present note, the authors have obtained certain expansions of Sylvester polynomials $\Psi_n(x)$ in terms of the Cesaro polynomials $g_n^{(m)}(x)$, Bernoulli polynomials $B_k(x)$, Bateman's polynomials $Z_n(x)$ and Generalized Bessel polynomials $\varphi_n(x)$. Generalizations of two of our main results are also considered.

Keywords: Sylvester polynomials, Cesaro polynomials, Bernoulli polynomials.

1. Introduction

Special functions, which are the solutions of certain differential equations play an important role in the formalism of Mathematical Physics and Biological Sciences and have applications in the areas of Heat conduction, Communication systems, Electro Magnetic theory, Quantum mechanics, non-linear wave propagating and probability theory, etc. and therefore provide a unique tool for developing simplified relativistic models of physical problems.

A vast Mathematical literature has been constructed and developed by Euler, Gauss, Hardy, Hermite, Ramanujan, and other classical authors. Several expansions of some classical Special functions and polynomials are seen in terms of other polynomials and Special functions. These types of results are not only the expansions but also the relations between Special functions/polynomials. and therefore are useful for developing the Special functions theory in different areas of pure and applied Mathematics. Also Several useful expansions of Special Functions in terms of other functions and polynomials are seen in the book¹.

Motivated by the above expansions, we, in the present note aim to give the expansions of Sylvester polynomials in terms of the Cesaro polynomials, Bernoulli polynomials, Bateman's polynomials and Generalized Bessel polynomials. Generalizations of our main results are also considered.

The Sylvester polynomials appears in Rainville (p 302, ex. 4)¹ as:

$$(1.1) \quad (1-t)^{-x} e^{xt} = \sum_{n=0}^{\infty} \Psi_n(x) t^n,$$

which may be rewritten as:

$$(1.2) \quad \sum_{n=0}^{\infty} \Psi_n(x) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(x)_k x^n t^{n+k}}{n! k!},$$

The Generating function for Cesaro polynomials is given by Agarwal² as:

$$(1.3) \quad \sum_{n=0}^{\infty} g_n^{(m)}(x) t^n = (1-t)^{-m-1} (1-xt)^{-1}$$

2. Preliminaries

The x^n term for Bernoulli Polynomials is given in Rainville (p. 302, ex. 1)¹ as:

$$(2.1) \quad x^n = \sum_{k=0}^n \frac{n! B_k(x)}{k! (n-k+1)!}$$

The x^n term for Bateman's Polynomials is given in Rainville (p.285, eqⁿ. 4)¹ as:

$$(2.2) \quad x^n = (n!)^2 \sum_{k=0}^n \frac{(-n)_k (2k+1) Z_k(x)}{(n+k+1)!}$$

The x^n term for Generalized Bessel Polynomials is given in Rainville (p. 294, eqⁿ. 7)¹ as:

$$(2.3) \quad x^n = n! \sum_{k=0}^n \frac{(-1)^k (c+2k) \psi_k(c, x)}{(n-k)! (c)_{n+k+1}}$$

Also, the following relations are given in Rainville¹:

$$(2.4) \quad n! = (1)_n$$

$$(2.5) \quad (\lambda)_{m+n} = (\lambda)_m (\lambda+m)_n$$

$$(2.6) \quad (\alpha)_{n-k} = \frac{(-1)^k (\alpha)_n}{(1-\alpha-n)_k}$$

$$(2.7) \quad (n-k)! = \frac{(-1)^k n!}{(-n)_k}$$

$$(2.8) \quad (-n-s)_s = \frac{(-1)^s (n+s)!}{n!}$$

$$(2.9) \quad {}_pF_q \left[\begin{matrix} \alpha_1, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_p)_n}{(\beta_1)_n \dots (\beta_q)_n} \frac{z^n}{n!}$$

3. Main Results

In this section, we have given total four theorems which are concerned with the expansion of Sylvester polynomials $\Psi_n(x)$ in series of four polynomials respectively Cesaro polynomials $g_n^{(m)}(x)$, Bernoulli polynomials $B_k(x)$, Bateman's polynomials $Z_n(x)$ and Generalized Bessel polynomials $\varphi_n(x)$:

Theorem 3.1: Expansion of Sylvester polynomials $\Psi_n(x)$ in terms of Cesaro polynomials $g_n^{(m)}(x)$:

$$(3.1) \quad \Psi_n(x) = \sum_{s=0}^n {}_3F_1 \left[x, (-n+s), -n; (m+2-n+s); -1 \right] \\ * \frac{(-n)_s (-m-1)_n g_s^{(m)}(x)}{(n!)^2 (m+2-n)_s}$$

Proof: We first find the x^n term for $g_n^{(m)}(x)$, which will be useful in the derivation of our main result:

Equation (1.3) may be rewritten as:

$$(1 - xt)^{-1} = (1 - t)^{-(-m-1)} \sum_{n=0}^{\infty} g_n^{(m)}(x) t^n$$

$$\text{or} \quad \sum_{n=0}^{\infty} x^n t^n = \sum_{n=0}^{\infty} \frac{(-m-1)_n t^n}{n!} \sum_{k=0}^{\infty} g_k^{(m)}(x) t^k$$

Replacing n by $(n - k)$ on the right-hand side we get:

$$\sum_{n=0}^{\infty} x^n t^n = \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{(-m-1)_{n-k}}{(n-k)!} g_k^{(m)}(x) t^n,$$

which on equating the coefficients of t^n on both sides, we get:

$$x^n = \sum_{k=0}^n \frac{(-m-1)_{n-k}}{(n-k)!} g_k^{(m)}(x)$$

or

$$x^n = \sum_{k=0}^n \frac{(-n)_k (-m-1)_n}{(m-n+2)_k n!} g_k^{(m)}(x)$$

Using the above equation our right-hand side of equation (1.2), we get:

$$\sum_{n=0}^{\infty} \Psi_n(x) t^n = \sum_{n,k=0}^{\infty} \frac{(x)_k t^{n+k}}{k! n!} \sum_{s=0}^n \frac{(-n)_s (-m-1)_n}{(m-n+2)_s n!} g_s^{(m)}(x)$$

$$\text{or} \quad \sum_{n=0}^{\infty} \Psi_n(x) t^n = \sum_{n,k,s=0}^{\infty} \frac{(x)_k (-n)_s (-m-1)_n g_s^{(m)}(x) t^{n+s+k}}{k! n! (m-n+2)_s n!}$$

Now, replacing n by $(n + s)$, on the right-hand side we get:

$$\sum_{n=0}^{\infty} \Psi_n(x) t^n = \sum_{n,k,s=0}^{\infty} \frac{(x)_k (-n-s)_s (-m-1)_{n+s} g_s^{(m)}(x) t^{n+s+k}}{k! (n+s)! (m+2-n-s)_s (n+s)!}$$

$$\text{or} \quad \sum_{n=0}^{\infty} \Psi_n(x) t^n = \sum_{n,k,s=0}^{\infty} \frac{(x)_k (-m-1)_{n+s} (n-m-2)! g_s^{(m)}(x) t^{n+s+k}}{k! (n+s)! (n-m-2+s)! n!}$$

Further replacing n by $(n - k)$, on the right-hand side we get:

$$\begin{aligned}\sum_{n=0}^{\infty} \Psi_n(x) t^n &= \sum_{n,s=0}^{\infty} \sum_{k=0}^n \frac{(x)_k (-m-1)_{n+s-k} (n-m-2-k)! g_s^{(m)}(x) t^{n+s}}{k! (n-k)! (n+s-k)! (n-m-2+s-k)!} \\ &= \sum_{n,s=0}^{\infty} \sum_{k=0}^n \frac{(x)_k (-n)_k (-n-s)_k (-m-1)_{n+s} (n-m-2)! g_s^{(m)}(x) t^{n+s}}{(-1)^k (m+2-n)_k k! (n+s)! n! (n-m-2+s)!}\end{aligned}$$

$$\begin{aligned}\text{or } \sum_{n=0}^{\infty} \Psi_n(x) t^n &= \sum_{n,s=0}^{\infty} {}_3F_1 \left[x, -n, (-n-s); (m+2-n); -1 \right] \\ &\quad * \frac{(-m-1)_{n+s} (n-m-2)! g_s^{(m)}(x) t^{n+s}}{n! (n+s)! (n-m-2+s)!}\end{aligned}$$

Now, replacing n by $(n - s)$, on the right-hand side we get:

$$\begin{aligned}\sum_{n=0}^{\infty} \Psi_n(x) t^n &= \sum_{n=0}^{\infty} \sum_{s=0}^n {}_3F_1 \left[x, (-n+s), -n; (m+2-n+s); -1 \right] \\ &\quad * \frac{(-m-1)_n (n-m-2-s)! g_s^{(m)}(x) t^n}{n! (n-s)! (n-m-2)!}\end{aligned}$$

$$\begin{aligned}\text{or } \sum_{n=0}^{\infty} \Psi_n(x) t^n &= \sum_{n=0}^{\infty} \sum_{s=0}^n {}_3F_1 \left[x, (-n+s), -n; (m+2-n+s); -1 \right] \\ &\quad * \frac{(-n)_s (-m-1)_n g_s^{(m)}(x) t^n}{(n!)^2 (m+2-n)_s},\end{aligned}$$

which on equating the coefficient of t^n on both sides, gives:

$$\Psi_n(x) = \sum_{s=0}^n {}_3F_1 \left[x, (-n+s), -n; (m+2-n+s); -1 \right] \frac{(-n)_s (-m-1)_n g_s^{(m)}(x)}{(n!)^2 (m+2-n)_s},$$

which proves Theorem 3.1.

Theorem 3.2: Expansion of Sylvester polynomials $\Psi_n(x)$ in terms of Bernoulli polynomials $B_k(x)$:

$$(3.2) \quad \Psi_n(x) = \sum_{s=0}^n {}_2F_0 \left[x; (-n-s-1); -; -1 \right] \frac{B_s(x)}{s! (n-s+1)!}$$

Proof: Using equation (2.1) in equation (1.2), we get:

$$\begin{aligned}\sum_{n=0}^{\infty} \Psi_n(x) t^n &= \sum_{n,k=0}^{\infty} \frac{(x)_k t^{n+k}}{k! n!} \sum_{s=0}^n \frac{n! B_s(x)}{s! (n-s+1)!} \\ &= \sum_{n,k=0}^{\infty} \sum_{s=0}^n \frac{(x)_k B_s(x) t^{n+k}}{k! s! (n-s+1)!}\end{aligned}$$

Replacing n by $(n + s)$, on the right-hand side we get:

$$\sum_{n=0}^{\infty} \Psi_n(x) t^n = \sum_{n,k,s=0}^{\infty} \frac{(x)_k B_s(x) t^{n+s+k}}{k! s! (n+1)!},$$

which, when n is replaced by $(n - k)$, the right-hand side yields:

$$\begin{aligned}
 \sum_{n=0}^{\infty} \Psi_n(x) t^n &= \sum_{n,s=0}^{\infty} \sum_{k=0}^n \frac{(x)_k B_s(x) t^{n+s}}{k! s! (n+1-k)!} \\
 &= \sum_{n,s=0}^{\infty} \sum_{k=0}^n \frac{(x)_k (-n-1)_k B_s(x) t^{n+s}}{k! (-1)^k s! (n+1)!} \\
 &= \sum_{n,s=0}^{\infty} {}_2F_0 [x; (-n-1); -: -1] \frac{B_s(x) t^{n+s}}{s! (n+1)!}
 \end{aligned}$$

Further, replacing n by $(n-s)$, on the right-hand side of the last equation we get:

$$\sum_{n=0}^{\infty} \Psi_n(x) t^n = \sum_{n=0}^{\infty} \sum_{s=0}^n {}_2F_0 [x; (-n-s-1); -: -1] \frac{B_s(x) t^n}{s! (n-s+1)!},$$

which on equating the coefficients of t^n on both sides, gives:

$$\Psi_n(x) = \sum_{s=0}^n {}_2F_0 [x; (-n-s-1); -: -1] \frac{B_s(x)}{s! (n-s+1)!}$$

which is the desired result (3.2) i.e. Theorem 3.2.

Theorem 3.3: Expansion of Sylvester polynomials $\Psi_n(x)$ in terms of Bateman's polynomials $Z_k(x)$:

$$(3.3) \quad \Psi_n(x) = \sum_{s=0}^n {}_3F_1 [x; (-n+s), (-n-s-1); (n)^2; 1] * \frac{(-n)_s n! (2s+1) Z_s(x)}{(n+s+1)!}$$

Proof: Using equation (2.2) in equation (1.2), we find:

$$\begin{aligned}
 \sum_{n=0}^{\infty} \Psi_n(x) t^n &= \sum_{n,k=0}^{\infty} \frac{(x)_k t^{n+k}}{k! n!} (n!)^2 \sum_{s=0}^n \frac{(-n)_s (2s+1) Z_s(x)}{(n+s+1)!} \\
 &= \sum_{n,k=0}^{\infty} \sum_{s=0}^n \frac{(x)_k (-n)_s (2s+1) n! Z_s(x) t^{n+k}}{k! (n+s+1)!}
 \end{aligned}$$

Replacing n by $(n+s)$, the right-hand side we get:

$$\begin{aligned}
 \sum_{n=0}^{\infty} \Psi_n(x) t^n &= \sum_{n,k,s=0}^{\infty} \frac{(x)_k (-n-s)_s (n+s)! (2s+1) Z_s(x) t^{n+s+k}}{k! (n+2s+1)!} \\
 &= \sum_{n,k,s=0}^{\infty} \frac{(x)_k (-1)^S \{(n+s)!\}^2 (2s+1) Z_s(x) t^{n+s+k}}{n! k! (n+2s+1)!}
 \end{aligned}$$

Further replacing n by $(n-k)$, the right-hand side we get:

$$\begin{aligned}
 \sum_{n=0}^{\infty} \Psi_n(x) t^n &= \sum_{n,s=0}^{\infty} \sum_{k=0}^n \frac{(x)_k (-1)^S \{(n+s-k)!\}^2 (2s+1) Z_s(x) t^{n+s}}{k! (n-k)! (n+2s+1-k)!} \\
 &= \sum_{n,s=0}^{\infty} \sum_{k=0}^n \frac{(x)_k (-n)_k (-n-2s-1)_k}{k! \{(n-s)_k\}^2} * \frac{(-1)^S \{(n+s)!\}^2 (2s+1) Z_s(x) t^{n+s}}{(n+2s+1)!} \\
 &= \sum_{n,s=0}^{\infty} {}_3F_1 [x, -n, (-n-2s-1); (-n-s)^2; 1] * \frac{(-1)^S \{(n+s)!\}^2 (2s+1) Z_s(x) t^{n+s}}{(n+2s+1)!}
 \end{aligned}$$

Now replacing n by $(n - s)$, the right-hand side we get:

$$\sum_{n=0}^{\infty} \Psi_n(x) t^n = \sum_{n=0}^{\infty} \sum_{s=0}^n {}_3F_1 \left[x; (-n + s), (-n - 2s - 1); (n)^2; 1 \right] \\ * \frac{(-n)_s n! (2s+1) Z_s(x) t^n}{(n+s+1)!},$$

which on equating the coefficient of t^n on both sides, gives:

$$\Psi_n(x) = \sum_{s=0}^n {}_3F_1 \left[x; (-n + s), (-n - s - 1); (n)^2; 1 \right] \\ * \frac{(-n)_s n! (2s+1) Z_s(x)}{(n+s+1)!},$$

which proves the result (3.3).

Theorem 3.4: Expansion of Sylvester polynomials $\Psi_n(x)$ in terms of Generalized Bessel polynomials $\psi_k(c, x)$, the following expansion exists:

$$(3.4) \quad \Psi_n(x) = \sum_{s=0}^n {}_3F_0 \left[x; (-n + s), (-n - c - s); -; 1 \right] \\ * \frac{(-1)^s (c+2s) \psi_s(c, x)}{(n-s)! (c)_{n+s+1}}$$

Proof: Using equation (2.3) in equation (1.2), and after similar steps of calculations as in the proofs of the above theorems, we get (3.4).

4. Generalizations

Here we have considered the generalizations of our two main results i.e. Theorem 3.1 and Theorem 3.2 as:

1. The Generalization of result (3.1) i.e. Theorem 3.1 is given by:

$$(4.1) \quad \Psi_n(x; a) = \sum_{s=0}^n {}_3F_1 \left[x; (-n + s), (1 - \lambda - n); -; \frac{-1}{a} \right] \\ * \frac{a^n (-n)_s (-m-1)_n g_s^{(m)}(\lambda, x)}{(\lambda)_n n! (m-n+2)_s},$$

where $\Psi_n(x; a)$ is Generalized Sylvester Polynomials defined by Agarwal and Manocha² as:

$$(4.2) \quad (1 - t)^{-x} e^{axt} = \sum_{n=0}^{\infty} \Psi_n(x; a) t^n$$

and $g_n^{(m)}(\lambda, x)$ is Generalized Cesaro polynomials defined by N. Özmen and E. Erkus-Duman⁴ as:

$$(4.3) \quad \sum_{n=0}^{\infty} g_n^{(m)}(\lambda, x) t^n = (1 - t)^{-m-1} (1 - xt)^{-\lambda}$$

2. The Generalization of result (3.2) i.e. Theorem 3.2 is given by:

$$(4.4) \quad \Psi_n(x; a) = \sum_{s=0}^n {}_2F_0 \left[x; (-s - m); -; \frac{-1}{a} \right] * \frac{(-1)^s a^n (-n)_s B_{n-s}^{(m-1)}(x)}{n! (s+m)!}$$

where $B_{n-s}^{(m-1)}(x)$ is Generalized Bernoulli polynomials defined in the paper⁵ by the following generating function:

$$(4.5) \quad \sum_{n=0}^{\infty} B_n^{(m-1)}(x; a) \frac{t^n}{n!} = \frac{t^m e^{xt}}{e^t - \sum_{h=0}^{m-1} \frac{t^h}{h!}}$$

6. Conclusions

We conclude by submitting that the Sylvester polynomials $\Psi_n(x)$, which is defined in terms of Hypergeometric functions ${}_2F_0$ has applications as Sylvester Matrix in Mathematical Physics and therefore its expansions in terms of certain expansions of Sylvester polynomials $\Psi_n(x)$ in terms of the Cesaro polynomials $g_n^{(m)}(x)$, Bernoulli polynomials $B_k(x)$, Bateman's polynomials $Z_n(x)$ and Generalized Bessel polynomials $\varphi_n(x)$. May of great importance further what we have given in this note seems to be interesting from the fact that the results are in terms of Generalized Hypergeometric functions. We have also generalized our main results and got expansions of Generalized Sylvester polynomials $\Psi_n(x; a)$ in terms of Generalized Cesaro polynomials $g_n^{(m)}(\lambda, x)$ and in terms of Generalized Bernoulli polynomials $B_{n-s}^{(m-1)}(x)$, which is the best part of our paper.

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