

A Note on Application of Fractional Integral Operators on Bessel Functions of First Kind

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Abstract: Two integral transforms involving Gaussian hypergeometric functions in the Kernel, which are generalization of classical Riemann-Liouville and Erdelyi-Kober fractional integral are considered and Applied on the Bessel functions of the first kind and thus two main results are derived, which are given in the form of two theorems. These two main results are obtained in terms of generalized wright functions, which are also expressed in terms of generalized hypergeometric functions, Corollaries & special cases are also considered.

Keywords: Bessel Functions of first kind, Gauss hypergeometric functions, Generalized Wright hypergeometric functions.

1. Introduction

Fractional calculus is that field of applied Mathematics, which deals with derivatives & Integral of arbitrary orders. The applications of fractional calculus are seen in various fields, including turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermal nuclear fusion, Non linear control theory, image Processing, non linear biological system, astrophysics etc.

In the recent past many researchers¹⁻³ are working on the application of the left and right sided generalized integral transforms defined and studied by Saigo². These Integral transforms are the generalization of the classical Riemann-Liouville & Erdelyi-Kober fractional integral operators and are defined as²:

$$(1.1) \quad (I_{0,+}^{\alpha,\beta,\eta} f)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_x^\infty (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta-\eta; \alpha; 1-\frac{t}{x}\right) f(t) dt$$

and

$$(1.2) \quad (I_{0,-}^{\alpha,\beta,\eta} f)(x) = \frac{1}{\Gamma\alpha} \int_x^\infty \frac{(t-x)^{\alpha-1}}{t^{\alpha+\beta}} {}_2F_1\left(\alpha+\beta-\eta; \alpha; 1-\frac{x}{t}\right) f(t) dt$$

where $\Gamma\alpha$ is the Euler Gamma Function⁴ and ${}_2F_1$ is the Hypergeometric function⁵.

when $\beta = -\alpha$, the equations (1.1) and (1.2) will reduce respectively to :

$$(1.3) \quad (I_{0,+}^\alpha f)(x) = \frac{1}{\Gamma\alpha} \int_0^x (x-t)^{\alpha-1} {}_2F_1 f(t) dt, \quad (x > 0)$$

$$(1.4) \quad (I_{0,-}^\alpha f)(x) = \frac{1}{\Gamma\alpha} \int_x^\infty (t-x)^{\alpha-1} f(t) dt, \quad (x > 0).$$

It is to note that equations (1.3) and (1.4) above are the left and right hand sided Riemann-Liouville fractional integral of order $\alpha \in \mathbb{C}$ given in the papers⁶⁻⁸. The Fractional Calculus now a days is one of the most rapidly growing subject of Mathematical Analysis. The fractional integral operator Involving various Special Functions has found significant importance and applications in various sub field of applicable Mathematical analysis. Some Applications are also seen in image Processing etc. In the last three decades, a number of workers like Love⁹, McBride¹⁰, Kalla^{11,12}, Kalla and Saxena¹³, Saigo¹⁴⁻¹⁵, Kilbas¹⁶, have studied the properties, applications different extension of various operators of Fractional Calculus. A account of fractional calculus operators along with their properties and application can be found in the research monographs by Kiryakova¹⁷, Miller and Ross¹⁸. The first application of Fractional Calculus was done due to Abel in the solution to the fractional problem. In Fractional Calculus, the fractional derivative are defined via fractional integral. The Computation of fractional derivatives and fractional integral of Special Functions of one and more variables is important from the point of view of the usefulness of these results in the evaluation of generalized Integrals and the solution of differential and integral equations. Motivated by these avenues of applications, a remarkable large number of fractional integral formulae involving a variety of Special Functions have been developed by many authors¹⁹⁻²⁰.

Motivated by the above recent applications of Fractional Calculus on various classical Special Functions and Polynomials, the Authors in the present note have applied the fractional integral operators defined by equations (1.1) and (1.2) above on the Bessel functions of first kind $J_\nu(z)$ defined as²¹:

$$(1.5) \quad J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left[\frac{z}{2}\right]^{v+2k}}{\Gamma(v+k+1).k!}$$

The Bessel functions of the first kind $J_n(x)$ are the solutions of the Bessel differential Equation:

$$(1.6) \quad x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - D^2)y = 0$$

which are non-Singular at the origin. These are sometimes also called cylinder function or cylindrical harmonic functions. The Bessel functions are playing an important role in a wide range of problems of Mathematical physics like problems of stochastic, Radio physics, hydro dynamics, atomic and nuclear physics, which led to various research workers, who are working in the field of Special Functions to explore various extensions and applications of these Bessel Functions.

In the recent past, various generalizations, extensions of Bessel Functions have been given by many Researchers²¹⁻²⁵ who are working in the field of Special Functions and their applications.

Bessel Functions of the first kind of order ϑ defined by G. N. Watson²⁶ occurs frequently in the problems like electrical engineer, finite elasticity, wave Mechanics, Mathematical physics and chemistry, where as the product of Bessel and Modified Bessel functions of the first kind appear frequently in Problems of statistical Mechanics and plasma physics²⁷. Very recently various researchers²⁶, are applying Fractional Calculus to the Bessel Functions and related Bessel functions of first kind and second kind and also on Modified Bessel functions.

The Generalized Bessel function of the first kind $w_{p,b,c}(z) = w_p(x)$ is defined for $z \in \mathbb{C}$, and $b, c, p \in \mathbb{C}, R(p) > -1$ by the following series:

$$(1.7) \quad w_{p,b,c}(z) = w_p(z) = \sum_{k=0}^{\infty} \frac{(-1)^k c^k}{k! \Gamma(p + \frac{(b+1)}{z} + k)} \left(\frac{z}{2}\right)^{2k+1},$$

where $\mathbb{C}$ denotes the set of complex numbers and Γ is the familiar Gamma Function.

The special cases of series (1.7) can be obtained as follows:

Case 1- if we put $b = c = 1$ in equation (1.7), we obtain the familiar Bessel function of the first kind²⁸, of order p for z , $p \in \mathbb{C}$ with $R(p) > -1$ defined and represented by the following expression²⁸:

$$(1.8) \quad J_p(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(p+k+1)} \left(\frac{z}{2}\right)^{2k+p}, \quad z \in \mathbb{C}$$

Case 2- putting $b = 1$ and $c = -1$ in series (1.7), we get the Modified Bessel function of the first kind of order p defined by:

$$(1.9) \quad I_p(z) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(p+k+1)} \left(\frac{z}{2}\right)^{2k+p}, \quad z \in \mathbb{C}$$

The series given by equation (1.9) is also a special case of Galue's Generalized Modified Bessel function²⁸ depending on parameters $a=0,1,2,\dots$ and $p>-1$, given as follows:

$$(1.10) \quad {}_aI_p(z) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(p+ak+1)} \left(\frac{z}{2}\right)^{2k+p}, \quad z \in \mathbb{C}$$

Case 3- letting $b = 2$ and $c = 1$ in series (1.7), we have the spherical Bessel function of the first kind of order p defined by²⁸,

$$(1.11) \quad I_p(z) = \sqrt{\frac{\pi}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(p+k+\frac{3}{2})} \left(\frac{z}{2}\right)^{2k+p}, \quad z \in \mathbb{C}$$

This paper is divided in two sections A and B. In the section A, the two main results are obtained in terms of Generalized Wright Function defined by Watson²⁶, which are given in the form of two theorems numbered as (3.1) and (3.2). Corollaries of the theorems are also considered. In the section B, we have represented the result obtained in section A in terms of Generalized hypergeometric function²³, which are also given in terms of two theorems (3.3) and (3.4). Corollaries of these theorems are also given.

2. Preliminaries

The following results will be used in the proof of our main theorems:

LEMMA I²⁹

$$(2.1) \quad (I_{0,+}^{\alpha,\beta,\eta} t^{\sigma-1})(x) = \frac{\Gamma(\sigma) \Gamma(\sigma+\eta-\beta) x^{\sigma-\beta-1}}{\Gamma(\sigma-\beta) \Gamma(\sigma+\alpha+\eta)}$$

where $\alpha, \beta, \eta \in \mathbb{C}$, $\operatorname{Re}(\alpha) > 0$ & $\operatorname{Re}(\sigma) > \max\{0, \operatorname{Re}(\beta-\eta)\}$

Equation (2.1) above has also the following two forms

$$(2.2) \quad (t^{\sigma-1} I_{0,+}^{\alpha})(x) = \frac{\Gamma(\sigma) x^{\sigma+\alpha-1}}{\Gamma(\sigma+\alpha)}, \quad \operatorname{Re}(\sigma) > 0$$

$$(2.3) \quad (I_{\eta^*,\alpha} t^{\sigma-1})(x) = \frac{\Gamma(\sigma+\eta) x^{\sigma+\alpha-1}}{\Gamma(\sigma+\alpha+\eta)}, \quad \operatorname{Re}(\sigma) > -\operatorname{Re}(\eta)$$

LEMMA II²⁹

$$(2.4) \quad (I_{-}^{\alpha,\beta,\eta} t^{\sigma-1})(x) = \frac{\Gamma(\beta-\sigma+1) \Gamma(\eta-\sigma+1) x^{\sigma-\beta-1}}{\Gamma(1-\sigma) \Gamma(\alpha+\beta+\eta-\sigma+1)}$$

where, $\alpha, \beta, \eta \in \mathbb{C}$, $\operatorname{Re}(\alpha) > 0$ & $\operatorname{Re}(\sigma) < 1 + \min[\operatorname{Re}(\beta), \operatorname{Re}(\eta)]$.

Equation (2.4) may also be expressed in the following two forms:

$$(2.5) \quad (I_{-}^{\alpha} t^{\sigma-1})(x) = \frac{\Gamma(1-\alpha-\sigma) x^{\sigma+\alpha-1}}{\Gamma(1-\sigma)}, \quad \operatorname{Re}(\alpha) < 1 - \operatorname{Re}(\sigma)$$

$$(2.6) \quad (K_{\eta,\alpha}^- t^{\sigma-1})(x) = \frac{\Gamma(\eta-\sigma+1)x^{\sigma-1}}{\Gamma(\alpha+\eta-\sigma+1)}, \quad \operatorname{Re}(\sigma) < 1 + \operatorname{Re}(\eta)$$

Gauss Hypergeometric Function²³

$$(2.7) \quad {}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!}$$

Generalized Wright Hypergeometric Function ${}_p\Psi_q(z)$ ²³:

$$(2.8) \quad {}_p\Psi_q(z) = {}_p\Psi_q \left[\begin{matrix} (a_i \alpha_i)_{1,p} \\ (b_j \beta_j)_{1,q} \end{matrix} \middle| z \right] = \sum_{k=0}^{\infty} \frac{\pi_{i=1}^p \Gamma(a_i + \alpha_i k) z^k}{\pi_{j=1}^q \Gamma(b_j + \beta_j k) K!}$$

$$(2.9) \quad \sum_{j=1}^q \beta_j - \sum_{i=1}^p \alpha_i > -1$$

Legendre duplication Formula⁴:

$$(2.10) \quad \Gamma 2z = \frac{2^{2z-1}}{\Gamma \pi} \Gamma z \Gamma \left(z + \frac{1}{2} \right)$$

Also from Rainville⁴ the following three results:

$$(2.11) \quad (z)_{2k} = 2^{2k} \left(\frac{z}{2} \right)_k \left(\frac{z+1}{2} \right)_k$$

$$(2.12) \quad \Gamma z + k = \Gamma z (z)_k$$

$$(2.13) \quad {}_p\Psi_q(a_1, a_2, a_3, \dots, a_p; b_1, b_2, b_3, \dots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k - (a_p)_k}{(b_1)_k - (b_q)_k} \frac{z^k}{K!}$$

3. Main Results

Theorem 3.1:

$$(3.1) \quad (I_{0,+}^{\alpha,\beta,\eta} t^{\sigma-1} J_v(t)) = \frac{x^{v+\sigma-\beta-1}}{2^v} \times {}_2\Psi_3 \left[\begin{matrix} (\sigma+v, 2), (\sigma+v+\eta-\beta, 2) \\ (\sigma+v-\beta, 2), (\sigma+v+\alpha+\eta, 2), (v+1, 1) \end{matrix} \middle| -\frac{x^2}{4} \right]$$

where $\alpha, \beta, \eta, \sigma, v, b, c \in \mathbb{C}$, $\operatorname{Re}(v+1) > 0$, $\operatorname{Re}(\alpha) > 0$ and

$$\operatorname{Re}(\sigma+v) > \max[0, \operatorname{Re}(\beta-\eta)].$$

Proof: Let us denote left hand side of equation (3.1) by P and then applying the equation (2.1) on P we get:

$$(3.2) \quad P = \left(I_{0,+}^{\alpha,\beta,\eta} t^{v+2k+\sigma-1} \right) \sum_{K=0}^{\infty} \left[\frac{(-1)^k \left(\frac{1}{2} \right)^{v+2k}}{\Gamma(v+k+1).k!} \right] (x)$$

{ using eqn (2.2) with σ replaced by $(v+2k+\sigma)$ }

$$= \frac{x^{v+\sigma-\beta-1}}{2^v} \sum_{k=0}^{\infty} \frac{\Gamma(v+\sigma+2k) \Gamma(\sigma+v+\eta-\beta+2k) \left(-\frac{x^2}{4} \right)^k}{\Gamma(v+\sigma-\beta+2k) \Gamma(v+\sigma+\eta+\alpha+2k) \Gamma(v+1+k).K!}$$

$$(3.3) \quad P = \frac{x^{v+\sigma-\beta-1}}{2^v} {}_2\Psi_3 \left[\begin{matrix} (\sigma+v, 2)(\sigma+v+\eta-\beta, 2) \\ (\sigma+v-\beta, 2)(\sigma+v+\eta+\alpha, 2)(v+1, 1) \end{matrix} \middle| -\frac{x^2}{4} \right], \quad (\text{using (2.9)})$$

which proves the Theorem (3.1), i.e. desired result (3.1).

Corollary 3.1: Substituting $\beta = 0$ in equation (3.1), we get the following result in terms of function ${}_1\Psi_2$:

$$(3.4) \quad \left(I_{0,+}^{\alpha} t^{\sigma-1} J_v(t) \right) (x) = \frac{x^{v+\sigma+\alpha-1}}{2^v} \times$$

$$\times {}_1\Psi_2 \left[\begin{matrix} (v+\sigma, 2) \\ (v+\sigma+\alpha, 2), (v+1, 1) \end{matrix} \middle| -\frac{x^2}{4} \right] \quad (\text{using (2.4)})$$

Theorem 3.2:

$$(3.5) \quad \left(I_{-}^{\alpha,\beta,\eta} t^{\sigma-1} J_v \left(\frac{1}{t} \right) \right) (x) = \frac{x^{\sigma-v-\beta-1}}{2^v} \times$$

$$\times {}_2\Psi_3 \left[\begin{matrix} (\beta-\sigma+v, 2), (\eta-\sigma+v+1, 2) \\ (1-\sigma+v, 2), (\alpha+\beta+\eta-\sigma+v+1, 2), (v+1, 1) \end{matrix} \middle| -\frac{1}{4x^2} \right]$$

where $\alpha, \eta, \sigma, v \in C$, $Re(v+1) > 0$ & $Re(\sigma-v) < 1 + \min\{R(\beta), R(\eta)\}$.

Proof: Let us denote left hand side of equation (3.5) by Q and then applying the equation (2.1) on Q we get:

$$Q = \sum_{K=0}^{\infty} \left[\frac{(-1)^k \left(\frac{z}{2} \right)^{v+2k}}{\Gamma(v+k+1).k!} \right] \left(I_{-}^{\alpha,\beta,\eta} t^{\sigma-2k-v-1} \right) (x)$$

$$= \frac{x^{v+\sigma-\beta-1}}{2^v} \sum_{k=0}^{\infty} \frac{\Gamma(\beta-\sigma+1+v+2k) \Gamma(\eta-\sigma+1+v+2k) \left(-\frac{1}{4x^2} \right)^k}{\Gamma(1-\sigma+v+2k) \Gamma(\alpha+\eta+\beta-\sigma+1+v+2k) \Gamma(v+k+1).K!}$$

(using Eqⁿ (2.5))

$$(3.6) \quad = \frac{x^{\sigma-\nu-\beta-1}}{2^\nu} {}_2\Psi_3 \left[\begin{matrix} (\beta-\sigma+\nu+1, 2)(\eta-\sigma+\nu+1, 2) \\ (1-\sigma+\nu, 2)(\alpha+\beta+\eta-\sigma+\nu+1, 2)(\nu+1, 1) \end{matrix} \middle| -\frac{1}{4x^2} \right] \quad (\text{using Eq}^n (2.9))$$

which proves the Theorem (3.2), i.e. desired result (3.5)

Corollary 3.2: Substituting $\beta = 0$ in equation (3.5), we get the following result in terms of function ${}_1\Psi_2$:

$$(3.7) \quad \left(I_{0,+}^\alpha t^{\sigma-1} J_\nu \left(\frac{1}{t} \right) \right) (x) = \frac{x^{\sigma-\nu+\alpha-1}}{2^\nu} \times {}_1\Psi_2 \left[\begin{matrix} (1+\nu-\alpha-\sigma, 2) \\ (1-\sigma+\nu, 2), (\nu+1, 1) \end{matrix} \middle| -\frac{1}{4x^2} \right].$$

Theorem 3.3:

$$(3.8) \quad \left(I_{0,+}^{\alpha,\beta,\eta} t^{\sigma-1} J_\nu(t) \right) (x) = \frac{x^{\nu+\sigma-\beta-1} \Gamma(\nu+\sigma) \Gamma(\nu+\sigma+\eta-\beta)}{2^\nu \Gamma(\nu+\sigma-\beta) \Gamma(\nu+\sigma+\alpha+\eta) \Gamma(\nu+1)} \times \\ \times {}_4F_5 \left[\begin{matrix} \frac{\nu+\sigma}{2}, \frac{\nu+\sigma+1}{2}, \frac{\nu+\sigma+\eta-\beta}{2}, \frac{\nu+\sigma+\eta-\beta+1}{2} \\ \frac{\nu+\sigma-\beta}{2}, \frac{\nu+\sigma-\beta+1}{2}, \frac{\nu+\sigma+\eta+\alpha}{2}, \frac{\nu+\sigma+\eta+\alpha+1}{2}, (\nu+1) \end{matrix} \middle| -\frac{x^2}{4} \right]$$

Proof: Let us denote the left hand side of equation (3.8) by R using equation (2.13) on R we get:

$$\begin{aligned} R &= \sum_{K=0}^{\infty} \frac{(-1)^k \left(\frac{1}{2} \right)^{\nu+2k}}{\Gamma(\nu+k+1) k!} \cdot \frac{\Gamma(\sigma+\nu) \Gamma(\nu+\sigma+\eta-\beta)}{\Gamma(\nu+\sigma-\beta) \Gamma(\nu+\sigma+\alpha+\eta)} \times \\ &\quad \times \frac{(\sigma+\nu)_{2k} (\sigma+\nu+\eta-\beta)_{2k}}{(\sigma+\nu-\beta)_{2k} (\nu+\sigma+\alpha+\eta)_{2k}} x^{\sigma+\nu-\beta+2k-1} \\ &= \sum_{K=0}^{\infty} \frac{x^{\sigma+\nu-\beta-1}}{2^\nu} \cdot \frac{\Gamma(\sigma+\nu) \Gamma(\nu+\sigma+\eta-\beta)}{\Gamma(\nu+1) \Gamma(\nu+\sigma-\beta) \Gamma(\nu+\sigma+\alpha+\eta)} \times \\ &\quad \times \frac{(\sigma+\nu)_{2k} (\sigma+\nu+\eta-\beta)_{2k}}{(\sigma+\nu-\beta)_{2k} (\nu+\sigma+\alpha+\eta)_{2k} (\nu+1)_{2k}} \left| -\frac{x^2}{4} \right|^k \frac{1}{k!} \\ &= \frac{x^{\sigma+\nu-\beta-1}}{2^\nu} \cdot \frac{\Gamma(\nu+\sigma) \Gamma(\nu+\sigma+\eta-\beta)}{\Gamma(\nu+\sigma-\beta) \Gamma(\nu+\sigma+\alpha+\eta) \Gamma(\nu+1)} \\ &\quad \times \sum_{K=0}^{\infty} \frac{\left(\frac{\nu+\sigma}{2} \right)_k \left(\frac{\nu+\sigma+1}{2} \right)_k \left(\frac{\nu+\sigma+\eta-\beta}{2} \right)_k \left(\frac{\nu+\sigma+\eta-\beta+1}{2} \right)_k}{(\nu+1)_k \left(\frac{\nu+\sigma-\beta}{2} \right)_k \left(\frac{\nu+\sigma-\beta+1}{2} \right)_k \left(\frac{\nu+\sigma+\alpha+\eta}{2} \right)_k \left(\frac{\nu+\sigma+\alpha+\eta+1}{2} \right)_k} \times \frac{\left(-\frac{x^2}{4} \right)^k}{k!} \\ &\quad (\text{using Eq}^n (2.12)) \end{aligned}$$

$$(3.9) \quad R = \frac{x^{v+\sigma-\beta-1} \Gamma(v+\sigma) \Gamma(v+\sigma+\eta-\beta)}{2^v \Gamma(v+\sigma-\beta) \Gamma(v+\sigma+\alpha+\eta) \Gamma(v+1)} \\ \times {}_4F_5 \left[\begin{matrix} \frac{v+\sigma}{2}, \frac{v+\sigma+1}{2}, \frac{v+\sigma+\eta-\beta}{2}, \frac{v+\sigma+\eta-\beta+1}{2} \\ \frac{v+\sigma-\beta}{2}, \frac{v+\sigma-\beta+1}{2}, \frac{v+\sigma+\eta+\alpha}{2}, \frac{v+\sigma+\eta+\alpha+1}{2} \end{matrix} \middle| \left(-\frac{x^2}{4} \right) \right],$$

(using Eqⁿ (3.1))

which proves theorem (4.1), i.e. desired result (3.8).

Corollary 3.3:

Substituting $\beta = -\alpha$ in equation (3.8), we get the following result in terms of function, ${}_2F_3$:

$$(3.10) \quad \left(I_{0,+}^\alpha t^{\sigma-1} J_v(t) \right)(x) = \frac{x^{v+\sigma+\alpha-1} \Gamma(v+\sigma)}{2^v \Gamma(v+\sigma+\alpha) \Gamma(v+1)} \times \\ \times {}_2F_3 \left[\begin{matrix} \frac{\sigma+v}{2}, \frac{\sigma+v+1}{2} \\ \frac{\sigma+v+\alpha}{2}, \frac{\sigma+v+\alpha+1}{2} \end{matrix} \middle| \left(-\frac{x^2}{4} \right) \right]$$

where $\alpha, \sigma, v, b, c \in C$, s.t. $Re(v+1) > 0, Re(\alpha) > 0$ & $Re(\sigma+v) > 0$.

Theorem 3.4:

$$(3.11) \quad \left(I_{0,-}^{\alpha,\beta,\eta} t^{\sigma-1} J_v \left(\frac{1}{t} \right) \right)(x) = \frac{\Gamma(\beta+\delta+1) \Gamma(\eta+\delta+1)}{\Gamma(\alpha+\beta+\eta+\delta+1) \Gamma(v+1)} \times \frac{x^{-\delta-\beta-1}}{2^v \Gamma(1+\delta)} \\ \times {}_4F_5 \left[\begin{matrix} \frac{\beta+\delta+1}{2}, \frac{\beta+\delta+2}{2}, \frac{\eta+\delta+1}{2}, \frac{\eta+\delta+2}{2} \\ \frac{1+\delta}{2}, \frac{2+\delta}{2}, \frac{\alpha+\beta+\eta+\delta+1}{2}, \frac{\alpha+\beta+\delta+\eta+2}{2} \end{matrix} \middle| \left(-\frac{1}{4x} \right) \right]$$

where $\alpha, \eta, \sigma, v \in C$ s.t. $Re(v+1) > 0, Re(\alpha) > 0$ and

$$Re(\sigma-v) < 1 + \min[Re(\beta), Re(\eta)]$$

Proof: Let us denote the left hand side of equation (3.11) by S using (2.13) on S we get:

$$S = \frac{x^{-\delta-\beta-1}}{2^v \Gamma(1+\delta)} \times \frac{\Gamma(\beta+\delta+1) \Gamma(\eta+\delta+1)}{\Gamma(\alpha+\beta+\eta+\delta+1) \Gamma(v+1)} \\ \times \sum_{k=0}^{\infty} \frac{(\beta+\delta+1)_{2k} (\eta+\delta+1)_{2k}}{(\alpha+\beta+\eta+\delta+1)_{2k} (1+\delta)_{2k} (v+1)_{2k} .k!} \left| \left(-\frac{1}{4x^2} \right) \right|^k \\ = \frac{x^{-\delta-\beta-1}}{2^v} \times \frac{\Gamma(\beta+\delta+1) \Gamma(\eta+\delta+1)}{\Gamma(\alpha+\beta+\eta+\delta+1) \Gamma(v+1)}$$

$$\times \sum_{K=0}^{\infty} \frac{\left(\frac{\beta+\delta+1}{2}\right)_{2k} \left(\frac{\beta+\delta+2}{2}\right)_k \left(\frac{\eta+\delta+2}{2}\right)_k}{\left(\frac{1+\delta}{2}\right)_k \left(\frac{\delta+2}{2}\right)_k \left(\frac{\alpha+\beta+\eta+\delta+1}{2}\right)_k \left(\frac{\alpha+\beta+\delta+\eta+2}{2}\right)_k (v+1)_k k!} \left|\left(\frac{-1}{4x^2}\right)^k\right|$$

(using Eqⁿ (2.12))

$$(3.12) \quad S = \frac{x^{-\delta-\beta-1}}{2^v} \times \frac{\Gamma(\beta+\delta+1)\Gamma(\eta+\delta+1)}{\Gamma(\alpha+\beta+\eta+\delta+1)\Gamma(v+1)}$$

$$\times {}_4F_5 \left[\begin{matrix} \frac{\beta+\delta+1}{2}, \frac{\beta+\delta+2}{2}, \frac{\eta+\delta+1}{2}, \frac{\eta+\delta+2}{2} \\ \frac{1+\delta}{2}, \frac{2+\delta}{2}, \frac{\alpha+\beta+\eta+\delta+1}{2}, \frac{\alpha+\beta+\delta+\eta+2}{2}, (v+1) \end{matrix} \right] \left|\left(\frac{-1}{4x^2}\right)^k\right|,$$

(using eqⁿ (3.1))

which Proves theorem (4.2), i.e. desired Result in terms of equationn (3.12).

Corollary 3.4:

Substituting $\beta = -\alpha$ in equation (3.12), we get the following result in terms of function, ${}_2F_3$:

$$(3.13) \quad \left(I_{0,-}^{\alpha,\beta,\eta} t^{\sigma-1} J_v \left(\frac{1}{t} \right) \right) = \frac{x^{-\delta+\alpha-1}}{2^v} \frac{\Gamma(-\alpha+\delta+1)}{\Gamma(v+1)\Gamma(1+\delta)} \times$$

$$\times {}_2F_3 \left[\begin{matrix} \frac{-\alpha+\delta+1}{2}, \frac{-\alpha+\delta+2}{2} \\ \frac{1+\delta}{2}, \frac{2+\delta}{2}, (v+1) \end{matrix} \right] \left| \left(-\frac{1}{4x^2} \right) \right|$$

4. Conclusions

Since Fractional derivatives and Integrals can be successfully used to formulate various phenomenon in physics, Biology, Electrical system, Thermodynamics etc, we in the present note have applied Riemann- Liouville and Erdelyi- Kober fractional integral operators on the Bessel Functions of first kind and obtain the results in terms of the Generalized wright hypergeometric function. Thereafter applying Certain formulae and by application of the some fractional Integral operators, we have obtained two interesting results in terms of the Generalized hypergeometric function. We have also discussed the corollaries of our main four Theorems, which give rise to other interesting results. These results may be of much importance to the researchers working on the application of Fractional calculus on various classical and Non- classical Special Functions.

The Authors will also wish to try further work on this paper in the coming days.

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