

Existence Theorems on Definite Almost Product Metric Manifolds

U. S. Negi and Preeti Chauhan

Department of Mathematics, H.N.B. Garhwal University, S.R.T. Campus

Badshahithaul, Tehri Garhwal – 249199, Uttarakhand, India

Email: usnegi7@gmail.com ; preetichauhan1011@gmail.com

(Received April 18, 2024, Accepted May 05, 2024)

Abstract: Negi¹ have derived some theorems on almost product and decomposable spaces, Negi at al.² has premeditated an analytic HP-transformation in almost Kaehlerian spaces. In this paper, we have find out existence theorems on definite almost product metric manifolds.

Keywords: Almost product manifolds, Kaehlerian manifolds, Riemannian spaces and Contact manifolds.

MSC 2020: 53C15, 53C55, 53B35

1. Introduction

A differentiable manifold is an almost product metric manifold (M^n) of class C^∞ which has a non-trivial tensor F_i^j with indices $i, j, k = 1, 2, \dots, n$, fulfilling the conditions³:

$$(1.1) \quad F_i^j F_j^k = \delta_i^k$$

$$(1.2) \quad g_{ij} F_h^i F_k^j = g_{hk}$$

If equation (1.1) is changed by $F_i^j F_j^k = -\delta_i^k$ then it is called an almost Hermitian manifold⁴ and satisfying the following two conditions:

$$(1.3) \quad \nabla_i F_j^i = 0,$$

$$(1.4) \quad \nabla_i (F_{jh}) = 0.$$

where $F_{jh} = F_j^i g_{ih}$ and ∇ is covariant derivative respecting Riemannian metric g_{ij} . Also, N_{ji}^h represents Nijenhuis tensor of F_i^j . In M^n , the following conditions are equivalents⁵⁻⁸:

$$(1.5) \quad \nabla_j F_i^h = 0,$$

$$(1.6) \quad \nabla_j F_{ih} + \nabla_i F_{jh} = 0,$$

$$(1.7) \quad (\delta_j^m \delta_i^l - F_j^m F_i^l) \nabla_m F_l^h = 0,$$

$$(1.8) \quad (\delta_j^m \delta_i^l + F_j^m F_i^l) \nabla_m F_l^h = 0,$$

$$(1.9) \quad F_{jl} + \nabla_j F_{lr} - \nabla_l F_{rj} = 0.$$

$$(1.10) \quad N_{ij}^h = F_i^l (\nabla_l F_j^h - \nabla_j F_l^h) - F_j^l (\nabla_l F_i^h - \nabla_i F_l^h).$$

$$(1.11) \quad ds^2 = g_{\lambda\mu}(x^\nu) dx^\lambda dx^\mu + g_{ab}(x^c) dx^a dx^b.$$

$$(1.12) \quad \left[\begin{array}{l} \left\{ \begin{array}{c} a \\ \lambda \quad \mu \end{array} \right\} = -\frac{1}{2} g^{ab} \partial_b g_{\lambda\mu}, \quad \left\{ \begin{array}{c} a \\ \lambda \quad b \end{array} \right\} = \frac{1}{2} g^{ac} \partial_\lambda g_{bc} \\ \left\{ \begin{array}{c} a \\ a \quad \lambda \end{array} \right\} = \frac{1}{2} g^{\lambda\gamma} \partial_a g_{\mu\gamma}, \quad \left\{ \begin{array}{c} a \\ a \quad b \end{array} \right\} = -\frac{1}{2} g^{\lambda\gamma} \partial_\lambda g_{ab} \end{array} \right]$$

$$(1.13) \quad \nabla_i F_{jk} + \nabla_j F_{ki} + \nabla_k F_{ij} = 0,$$

This is identified that positive Riemannian metric g_{ij} in a new system of allotment accordingly definite almost product metric manifold⁹.

2. Existence theorems on definite Almost Product metric Manifolds

Theorem 2.1: *The necessary and sufficient conditions of a differentiable manifold M^n to be definite almost product metric manifold are following:*

$$(2.1) \quad \text{(i). } N_{ij}^h - N_{ij}^l F_l^h = 0 \quad \text{or} \quad N_{ij}^h + N_{ij}^l F_l^h = 0$$

$$(2.2) \quad \text{(ii). } T_{jih} = -T_{ijh}.$$

$$(2.3) \quad \text{(iii). } \nabla_l F_j^l = 0.$$

$$(2.4) \quad \text{(iv). } M_{ij}^j = 0.$$

$$(2.5) \quad \text{(v). } F_j^l M_{li}^h + F_i^l M_{lj}^h = 0.$$

Proof: We have proved the following:

(i) If we put:

$$(2.6) \quad M_{ij}^h = \nabla_i F_j^h + F_i^l F_j^m \nabla_l F_m^h + F_i^l (\nabla_l F_j^h - \nabla_j F_l^h),$$

Then $N_{ij}^h - N_{ij}^l F_l^h = 0$ if and only if $M_{ij}^h = M_{ji}^h$,

(ii). Here, putting: $T_{jih} = \nabla_j F_{ih} + F_j^m F_i^l \nabla_m F_{lh}$ and substitute (1.13) in the expression of T_{jih} , we get:

$$\begin{aligned} T_{jih} &= -(\nabla_i F_{hj} + \nabla_h F_{ji}) - F_j^m F_i^l (\nabla_l F_{hm} + \nabla_h F_{ml}) - (\nabla_i F_{hi} + F_j^l F_i^m \nabla_m F_{hl}) \\ &= -T_{ijh}. \end{aligned}$$

because of

$$(2.7) \quad \nabla_h F_{ji} + F_j^m F_i^l \nabla_h F_{ml} = 0,$$

which follows from $F_i^l F_l^h = \delta_i^h$ by covariant differentiation ∇_j , and then contracting with F_l^i . Conversely:

$$(2.8) \quad T_{jih} = \nabla_j F_{ih} - F_j^m F_h^l \nabla_m F_{il} = 2\nabla_j F_{hi} - T_{jhi}, \text{ i.e.} \\ 2\nabla_j F_{hi} = T_{jih} + T_{jhi}.$$

From which we have:

$$(2.9) \quad 2(\nabla_j F_{hi} + \nabla_h F_{ij} + \nabla_i F_{jh}) = T_{jih} + T_{jhi} + T_{hji} + T_{hij} + T_{ihj} + T_{ijh}.$$

From (2.2) in (2.4), then almost Hermitian manifold gratifying form equivalent to (1.13) is almost Kaehlerian manifold¹⁰.

$$(iii). \text{ We have: } g^{ji} T_{ijh} = g^{ji} \nabla_j F_{ih} + g^{ji} F_j^m F_h^l \nabla_m F_{lh} \\ = 2 g^{ji} \nabla_j F_{ih} = 2 \nabla_j F_h^j$$

from (1.6), we obtain: $\nabla_j F_h^j = -\nabla_j F_h^j \Rightarrow \nabla_j F_h^j = 0$.

(iv). From $\nabla_k (F_j^l F_l^i) = 0$, we have $F_l^m \nabla_k F_m^l = 0$. On the other hand, as $F_l^i = \text{constant}$, we get $\nabla_j F_l^i = 0$. Thus $M_{ji}^l = -F_j^l \nabla_m F_l^m$.

Then, F_j^l is non-singular.

(v). We first note that from $F_j^l F_{li} = g_{ji}$ then:

$$(2.10) \quad F_j^l \nabla_k F_{li} + F_i^l \nabla_k F_{jl} = 0.$$

Now, from (2.6), we have:

$$(2.11) \quad g_{hk} (F_j^l M_{li}^h + F_i^l M_{lj}^h) = F_j^l (\nabla_l F_{ik} + \nabla_i F_{lk}) + F_i^l (\nabla_j F_{lk} + \nabla_l F_{jk}).$$

If $F_j^l M_{li}^h + F_i^l M_{lj}^h = 0$ and from (2.10) and (2.11), we have the following:

$$(2.12) \quad F_j^l (\nabla_l F_{ik} + \nabla_i F_{lk} + \nabla_k F_{li}) = -F_i^l (\nabla_j F_{lk} + \nabla_l F_{jk} + \nabla_k F_{jl}).$$

From (2.12), $P_{jik} = -P_{ijk}$. But it is evident that $P_{jik} = P_{jki}$, therefore $P_{jik} = 0$ as (1.13) pursues F_i^j is non-singular and converse is apparent.

Theorem 2.2: *The necessary and sufficient condition of an almost product metric manifold (M^n) gained as of a contact manifold $(\bar{M}^n)$ content the condition $\nabla_l F_j^l = 0$. and $\nabla_i F_{jk} + \nabla_j F_{ki} + \nabla_k F_{ij} = 0$, then ξ^i is a Killing vector field.*

Proof: Consider a $(2m + 1)$ -dimensional differentiate manifold $(\bar{M}^n)$ by contact formation η , satisfying $\eta \wedge (d\eta)^n \neq 0$, that there exists a (ϕ, ξ, η, g) formation with³:

$$\eta = \eta_i dx^i, \quad d\eta = \phi = \frac{1}{2} \phi_{ij} dx^i \wedge dx^j; \quad (\phi_{ij} = \partial_i \eta_j - \partial_j \eta_i, \quad \partial_i \eta_j = \frac{\partial \eta_j}{\partial x^i}).$$

Conditioning other closely, there exist a vector field ξ^i , ϕ_i^j and g_{ij} such that:

$$\text{rank } |\phi_i^j| = 2m, \quad \phi_i^j \xi^j = 0, \quad \phi_j^i \eta_i = 0, \quad \xi^j \eta_j = 1, \quad \phi_j^i \phi_k^j = -\delta_k^i + \xi^i \eta_k,$$

$$g_{ij} \xi^j = \eta_i, \quad g_{ij} \phi_h^i \phi_k^j = g_{hk} - \eta_h \eta_k, \quad \phi_{ij} = g_{ij} \phi_i^h = \nabla_i \eta_j - \nabla_j \eta_i.$$

Therefore obvious that:

$$(2.13) \quad (\nabla_j \eta_k) \xi^k = 0,$$

and it is also known that¹¹:

$$(2.14) \quad \nabla_i \xi^i = 0,$$

$$(2.15) \quad (\nabla_k \eta_j) \xi^k = 0,$$

$$(2.16) \quad F_i^j = (2 \xi^j \eta_i - \delta_i^j).$$

From the identity (1.1) and (1.2), it is a M^n from the given $\bar{M}^n$.

Next, from (2.16) we have:

$$\nabla_h F_i^j = 2(\nabla_h \xi^j) \eta_i + 2 \xi^j (\nabla_h \eta_i).$$

So, by (2.14) and (2.15) we have:

$$\nabla_j F_i^j = 2(\nabla_j \xi^j) \eta_i + 2 \xi^j (\nabla_j \eta_i) = 0.$$

Next, as $F_{ij} = 2\eta_i \eta_j - g_{ij}$ then we find:

$$\frac{1}{2} \nabla_h F_{ij} = (\nabla_h \eta_i) \eta_j + \eta_i (\nabla_h \eta_j),$$

from which it follows:

$$(2.17) \quad \frac{1}{2} (\nabla_h F_{ij} + \nabla_i F_{jh} + \nabla_j F_{hi}) = \eta_h (\nabla_i \eta_j + \nabla_j \eta_i) \\ + \eta_i (\nabla_h \eta_j + \nabla_j \eta_h) + \eta_j (\nabla_i \eta_h + \nabla_h \eta_i).$$

Then, applying (1.13) and comparing ξ^h with the equation (2.17), we get:

$$(2.18) \quad (\nabla_i \eta_j + \nabla_j \eta_i) + \eta_i (\xi^h \nabla_k \eta_j + \xi^h \nabla_j \eta_k) + \eta_j (\xi^h \nabla_i \eta_k + \xi^h \nabla_k \eta_i) = 0.$$

Taking account of (2.13) and (2.15) we have from the above relation the following:

$$(2.19) \quad \nabla_i \eta_j + \nabla_j \eta_i = 0,$$

that is, ξ^i is Killing vector field and converse is apparent from (2.17).

Theorem 2.3: In a compact orientable M^n gratifying $\nabla_l F_j^l = 0$, if an insignificant projective transmutation on similar period drops F_i^j invariable, then this is an isometry.

Proof: The Lie derivative ($\mathcal{L}_v$) respecting an insignificant transmutation v^i and differentiable manifold (M^n) convinced (2.3). We have known that¹²:

$$(2.20) \quad \mathcal{L}_v \nabla_j F_i^h - \nabla_j \mathcal{L}_v F_i^h = F_i^l \mathcal{L}_v \left\{ \begin{matrix} h \\ j \ l \end{matrix} \right\} - F_l^h \mathcal{L}_v \left\{ \begin{matrix} l \\ j \ i \end{matrix} \right\}.$$

Now, reduce through j and then formulate from (2.3), we get:

$$(2.21) \quad -\nabla_l \mathcal{L}_v F_i^l = F_i^m \mathcal{L}_v \left\{ \begin{matrix} l \\ l \ m \end{matrix} \right\} - F_m^l \mathcal{L}_v \left\{ \begin{matrix} m \\ l \ i \end{matrix} \right\}.$$

On similar period, an insignificant projective transmutation v^i drops F_i^j invariable, then we has:

$$(2.22) \quad \mathcal{L}_v \left\{ \begin{matrix} h \\ j \ i \end{matrix} \right\} = \delta_j^h \psi_i + \delta_i^h \psi_j,$$

$$(2.23) \quad \mathcal{L}_v F_i^h = 0.$$

Substitute these two relations in (2.21) we have:

$$(2.24) \quad (n F_i^m - \delta_i^m F_l^l) \Psi_m = 0.$$

But we write this:

$$(2.25) \quad (n F_i^m - \delta_i^m F_l^l)(n F_m^t + \delta_m^t F_l^l) = \delta_i^t \{n^2 - (F_l^l)^2\} \neq 0.$$

As $(F_l^l)^2 = (n - 2p)^2 < n^2$. Thus $n F_i^m - \delta_i^m F_l^l$ is non-singular and we have $\Psi_m = 0$, from which and (2.22) we have:

$$(2.26) \quad \Psi_m = \frac{1}{n+1} \nabla_m \nabla_l v^l = 0.$$

Thus, $\nabla_l v^l = \text{constant}$ and $\mathcal{L}_v \left\{ \begin{smallmatrix} h \\ j \ i \end{smallmatrix} \right\} = 0$.

Also, if the space is compact orientable then using Green's theorem, we find:

$$\nabla_l v^l = 0 \quad \text{and} \quad \mathcal{L}_v \left\{ \begin{smallmatrix} h \\ j \ i \end{smallmatrix} \right\} = 0.$$

Hence theorem is proved.

Theorem 2.4: *In a compact orientable M^n gratifying $\nabla_l F_j^l = 0$, if an insignificant conformal transmutation on similar period drops F_i^j invariable, then this is an isometry.*

Proof: We have known that¹³:

$$(2.27) \quad \mathcal{L}_v \left\{ \begin{smallmatrix} h \\ j \ i \end{smallmatrix} \right\} = \frac{1}{2} g^{hn} (\nabla_j \mathcal{L}_v g_{ni} + \nabla_i \mathcal{L}_v g_{jn} - \nabla_n \mathcal{L}_v g_{ji}).$$

Substitute (2.23) and (2.27) in (2.21) we have:

$$(2.28) \quad F_i^m g^{ln} \nabla_m \mathcal{L}_v g_{ln} - F^{nl} \nabla_i \mathcal{L}_v g_{ln} = 0.$$

From $\mathcal{L}_v g_{ln} = \nabla_l v_n + \nabla_n v_l = 2\phi g_{ln}$, we have $\nabla_i \mathcal{L}_v g_{ln} = 2\phi_i g_{ln}$, substituted this formula into (2.28) we get:

$$(2.29) \quad F_i^m \nabla_m \nabla_l v^l - \phi_i F_l^l = 0.$$

Since $\phi = (1/n) \nabla_l v^l$ and $\phi_i = (1/n) \nabla_i \nabla_l v^l$, the above equation is written as

$$(2.30) \quad \left(F_i^m - \frac{1}{n} \delta_i^m F_t^t \right) \nabla_m \nabla_l v^l = 0.$$

From (2.30), we can get the proof of theorem 2.4.

Theorem 2.5: *In an almost product metric manifold M^n satisfying (2.3), then*

$$(2.31) \quad (\nabla_h F^{ji}) (\nabla_j F_i^h) = \bar{H} - R,$$

where $R_{jl} = R_{hjl}^h$, $R = g^{lj} R_{jl}$, $\bar{H}_{ji} = R_{hji} F^{lh}$ and $\bar{H} = F^{ji} \bar{H}_{ji}$.

Proof: We have reduced by h, k of following equation¹⁰:

$$(2.32) \quad \nabla_k \nabla_j F_i^h - \nabla_j \nabla_k F_i^h = R_{kjl}^h F_i^l - R_{kji}^l F_l^h,$$

and then make use of (2.3), we get:

$$(2.33) \quad \nabla_i \nabla_j F_i^l = R_{jl} F_i^l - \bar{H}_{ji}.$$

Differentiate $F^{ji} F_{ih} = \delta_h^j$ with ∇_j , then we have:

$$(2.34) \quad F^{ji} \nabla_j F_{ih} = 0.$$

using (2.3) in (2.34), we obtain:

$$(2.35) \quad (\nabla_h F^{ji})(\nabla_j F_i^h) = -F^{ji}(\nabla_h \nabla_j F_i^h).$$

Also, as $F_i^l F^{ji} = g^{lj}$ and use equation (2.33), then we get:

$$(2.36) \quad (\nabla_h F^{ji})(\nabla_j F_i^h) = -F^{ji}(R_{jl} F_i^l - \bar{H}_{ji}) = \bar{H} - R.$$

Thus, theorem is proved.

References

1. U. S. Negi; Theorems on Almost product and decomposable spaces, *Aryabhatta J. Math. Inform.* 9-1 (2017), 105-110.
2. U. S. Negi, Trishna Devi and M.S. Poonia; An analytic HP-transformation in almost Kaehlerian spaces. *Aryabhatta Journal of Mathematics & informatics*, 11-1 (2019), 103-108.
3. S. Sasaki, and Y. Hatakeyama; On differentiable manifolds with contact metric structures, *Jour. of Math. Soc. Japan*, 14 (1962), 249-271.
4. S. Sasaki; On differential geometry of tangent bundles of Riemannian manifold, *Tohoku Math. Jour.*, 10 (1958), 338-354.
5. Y. C. Wong; A note on complementary subspaces in a Riemannian space, *Bull. Amer. Math. Soc.*, 49 (1943), 120-125.
6. S. Tachibana; On almost-analytic vectors in certain almost Hermitian manifolds, *Tohoku Math. Jour.*, 11(1959), 351-363.
7. S. Tachibana; Some theorems on locally product Riemannian spaces, *Tohoku Math. Jour.*, 12 (1960), 281-293.
8. S. Tachibana and M. Okumura; On the almost-complex structure of tangent bundles of Riemannian spaces, *Tohoku Math. Jour.*, 14(1962), 156-161.
9. B. Reinhart; Foliated manifold with bundle-like metrics, *Ann. of Math.*, 69 (1959), 119-131.
10. S. Koto; Some theorems on almost Kaehlerian space, *Jour. of Math. Soc. Japan*, 12 (1960), 422-433.
11. C. J. Hsu; On some structures which are similar to the quaternion structure, *Tohoku Math. Jour.*, 12 (1960), 403-428.
12. K. Yano; Affine connections in an almost product space, *Kodai Math. Sem. Rep.*, 11 (1959), 1-24.
13. K. Yano; Conformally separable quadratic differential forms, *Proc. Imp. Acad. Tokyo*, 16 (1940), 83-86.