

# A Study on Mathematical Modelling in Two Phase Arterial Blood Flow during Major $\beta$ Thalassemia (Cooley anemia)

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**Abstract:** Major  $\beta$  Thalassemia is a disease of the red blood cells that contains haemoglobin. Haemoglobin carries oxygen to the tissues. In this case RBC can't carry oxygen. We introduced the non-Newtonian two phase arterial Blood Flow model. Blood is considered as a homogeneous mixture of red blood cells and plasma that obeys the power law. We put the equation of continuity and motion into tensorial form. We obtained a correlation between haematocrit and pressure drop that predicate blood flow fluctuation. We also collected pathological data from Thalassemia patients and graphically studied blood flow variations. The solution methods adopted are both analytical and numerical.

**Keywords:** Two phase arterial blood flow, arterial blood flow in case of Cooley anemia, Non-Newtonian, Power law model, haematocrit, blood pressure drop.

## 1. Introduction

Major  $\beta$  Thalassemia is an inherited blood disorder. It causes the body to make less haemoglobin. Dr. Cooley first reported Major  $\beta$  Thalassemia in Detroit in 1925, leading to the name Cooley's anemia<sup>1,2</sup>. It is most common in Latin Americans, Middle Easterners and Asians. In the presence of the haemoglobin  $\beta$ -globins chain, haemoglobin carries oxygen to the tissues. Under such circumstance, red blood cells cannot transport oxygen efficiently<sup>2,3,4</sup>.

In our proposed model, the blood has been considered as a homogenous mixture of two phase; the first is the liquid phase of plasma and the second is that of RBC (Red blood cells, liquid packets in a semi permeable membrane

capable of changing its shape and size) with density higher than that of plasma. Flow pattern has been taken steady and axial symmetry with equal velocity of both phases<sup>5,6</sup>.

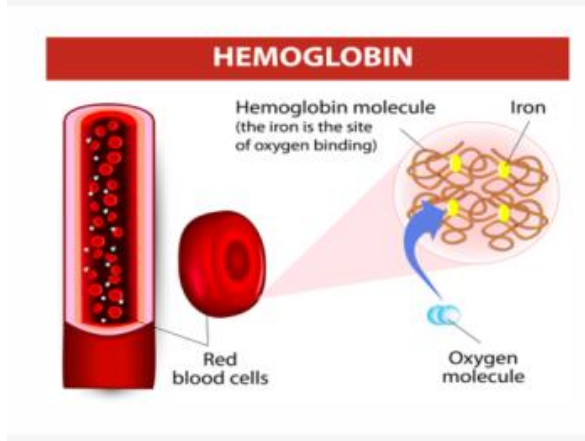


**Fig. 1:** oxygen in Red blood cells<sup>6</sup>

Mathematically when shear stress occurs viscosity generates, then there is a linear relationship between stress and strain rate i.e.  $\tau^{lk} = \eta_m e^{lk}$ . In narrow vessels like arteries, blood shows non-Newtonian nature, i.e.  $\tau^{lk} = \eta_m (e^{lk})^n$  and  $n \neq 1$ , this happens due to higher concentration of haematocrit which is made of haemoglobin<sup>7</sup>.

## 2. Real Model

### 2.1 Choice of frame of Reference

We have to select a frame of reference for mathematical modeling of the state of a moving blood keeping in view the difficulty and generality of the problem of blood flow. We select generalized three-dimensional orthogonal curvilinear co-ordinate system, briefly prescribed as  $E^3$ , called as 3-dim Euclidean space. We interpret the quantities related to blood flow in tensorial form which is comparatively more realistic<sup>8</sup>. The biophysical laws thus expressed fully hold good in any co-ordinate system, which is compulsion for the truthfulness of the law Now, let the co-ordinate axes be  $OX^i$  where  $O$  is origin and superscript  $i = 1, 2, 3$  let  $X^i$  be the co-ordinates of any point P in space. The mathematical description of the state if a moving blood is affected by means of functions which give the distribution of the blood velocity  $V^k = V^k(X^i, t)$ ,  $k = 1, 2, 3$  and of any two thermodynamic quantities pertaining to the blood, for instance the pressure  $p = p(X^i, t)$  and the density  $\rho = \rho(X^i, t)$ .<sup>9,10</sup>

## 2.2 Choice of parameters

Blood is Non-Newtonian fluids. If the relation between stress and strain rate are linear the flow is Newtonian otherwise Non-Newtonian.

The constitutive equations for fluids

$$(2.1) \quad \tau = \eta e^n.$$

If  $n = 1$  then the nature of fluid is Newtonian and if  $n \neq 1$  then the nature of fluid is non-Newtonian. Where  $\tau$  is denoted by stress.  $e$  is denoted by strain rate,  $\eta$  is denoted by viscosity and  $n$  is the parameter (other than dependent and independent variable) depends upon nature of fluid. In present study there are five parameters are used but three components of velocity  $v^k$ , blood pressure  $P$  and density  $\rho$ .<sup>5</sup>

## 2.3 Boundary conditions are as follows:

- The velocity of blood flow on the axis of arteries at  $r = 0$  will be maximum and finite, say  $v_0 = \text{maximum velocity}$ ,  $v = v_0$  then  $v_0 = 0$ .
- The velocity of blood flow on the wall of artery at  $r = R$ , where  $R$  is the radius of artery will be zero. This condition is well known as no-slip condition  $v = 0$  at  $r = R$ .

## 2.4 Two Phase Description

Blood is a complex fluid consisting of particulate corpuscles suspended in a non-Newtonian fluid. The particulate solids are red blood cells (RBCs), white blood cells (WBCs) and platelets. 55% of the plasma and 45% of the blood cells in a whole blood and approximately 98% of RBCs in 45% of blood cells and there are a few parts (approximately 2%) of the other cells. Which are ignorable, so one phase of the blood is plasma and 2nd phase of blood is RBCs.<sup>9</sup>

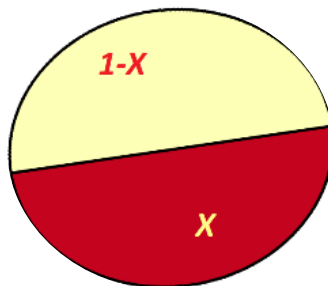


**Fig. 2:** Blood cells in unit volume

Let the volume portion covered by blood cells in unit volume be  $x$ ,  $x$  is replaced by  $\frac{H}{100}$ , where  $H$  is the hematocrit then the volume portion covered by the plasma will be  $(1 - X)$ .

### 3. Mathematical Formulation

Blood is not an ideal fluid, it is in liquid form. Whenever the hematocrit increases, the effective viscosity of blood flowing in the arteries proximate to the heart depend upon the strain rate. In this situation, the blood flow becomes Non-Newtonian. In this model the concentration of red blood cells becomes high. In this situation the constitutive equation of blood is as follows:<sup>5,11,12</sup>

$$(3.1) \quad \tau^{ij} = -pg^{ij} + \eta_m (e^{ij})^n = -pg^{ij} + T^{ij}.$$

#### 3.1 Low of Conservation of mass:

Blood flow must obey the principle of conservation of mass because there is no source and sink. Total flow is constant across all parts of circulatory system. On the other hand, we say that “blood inflows on the circulatory system are equal to the blood outflows from the circulatory system”.

#### 3.2 Mathematical Modeling:

Equation of continuity for power law flow will be as follows:

$$(3.2) \quad \frac{1}{\sqrt{g}(\sqrt{g}v^i)} = 0.$$

Equation of motion will be as follows:

$$(3.3) \quad \rho_m \frac{\partial v^i}{\partial t} + (\rho_n v_i) v^i_{,j} = \tau^{ij}_{,j},$$

where  $\tau^{ij}$  is taken from constitutive equation of per law flow.  $\rho_m = X\rho_c + (1-X)\rho_p$  is the density of blood and  $\eta^m = X\eta_c + (1-X)\eta_p$  is the viscosity of mixture of the blood<sup>8</sup>.  $X = H/100$  is volume ratio of the blood cell;  $H$  is the Hematocrit. Other symbols have their usual meanings<sup>9</sup>. Since the blood vessels are cylindrical, the above governing equations have to be transformed into cylindrical co-ordinates. Now we have to transform the equations in cylindrical form. As we know, for cylindrical coordinates<sup>13-16</sup>

$$X^1 = r, \quad X^2 = \Theta, \quad X^3 = z.$$

Matrix of metric tensor in cylindrical co-ordinates is as follows:

$$g = |g_{ij}| = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = r^2,$$

while matrix of conjugate matrix tensor is as follow:

$$[g^{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

whereas the Christoffel symbol of 2<sup>nd</sup> kind are as follow:

$$\left\{ \begin{matrix} 1 \\ 2 \end{matrix} \right\} = -r, \left\{ \begin{matrix} 2 \\ 2 \end{matrix} \right\} = \left\{ \begin{matrix} 2 \\ 1 \end{matrix} \right\} = \frac{1}{r}$$

Remaining others is zero. Relation between contra variant physical components of the blood flow will be as follows

$$\sqrt{g_{11}}V^1 = V_1 \Rightarrow V_r \Rightarrow V^1$$

$$\sqrt{g_{22}}V^2 = V_2 \Rightarrow V_\theta \Rightarrow V^2$$

$$\sqrt{g_{33}}V^3 = V_3 \Rightarrow V_z \Rightarrow V^3$$

Again the physical components of  $p_{,j}g^{ij}$  is  $-\sqrt{g_{11}}p_{,j}g^{ij}$ . The matrix of physical components of shearing stress-tensor  $\tau^{ij} = \eta_m(e^{ij})^n = \eta_m(e^{ij})^n = \eta_m(g^{ik}V_{,k}^i + g^{ik}V_{,k}^j)^n$  will be as follows

$$\begin{bmatrix} 0 & 0 & \eta_m \left( \frac{dV}{dt} \right)^n \\ 0 & 0 & 0 \\ \eta_m \left( \frac{dV}{dt} \right)^n & 0 & 0 \end{bmatrix}$$

The covariant derivative of  $\tau^{ij}$

$$\tau^{ij}_{,j} = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g} T^{ij}}{\partial X^j} + \left\{ \begin{matrix} i \\ j \end{matrix} \right\} \tau^{ij}$$

Keeping in view the above fact the governing tensorial equation can be transformed into cylindrical from which are as follows; the equation of continuity

$$\frac{\partial v}{\partial z} = 0$$

### 3.3 Equation of motion

$r$  –Component

$$-\frac{\partial p}{\partial r} = 0$$

$\theta$  – Component,

$$0 = 0$$

$z$  – Component

$$0 = -\frac{\partial p}{\partial z} + \frac{\eta_m}{r} \left[ r \left( \frac{\partial V_z}{\partial r} \right)^n \right]$$

Here, this fact has been taken in view that the blood flow is axially Symmetric in arteries concerned, *i.e.*

$$V_\theta = 0 \text{ and } V_r = 0$$

$V_z$  And  $p$  do not depend upon  $\theta$ . Also the blood flow steadily, i.e.

$$\frac{\partial p}{\partial t} = \frac{\partial V_r}{\partial t} = \frac{\partial V_\theta}{\partial t} = \frac{\partial V_z}{\partial t} = 0.$$

#### 4. Solution

On integration the equation of continuity we get

$$V_z = V_r \quad \text{as } V \text{ does not depend upon } \theta$$

The integrating of equation of motion yields:

$$P = p(z), \quad \text{since } p \text{ does not depend upon } \theta.$$

Now, with the help of equations  $v_z = v(r)$  and  $P = p(z)$ , the equation of motion converts in the following form:

$$(4.1) \quad 0 = -\frac{dp}{dr} + \frac{\eta_m}{r} \frac{d}{dr} \left( r \left( \frac{dv}{dr} \right)^n \right)$$

The pressure gradient  $-\left(\frac{dp}{dz}\right) = P$  of blood flow in the arteries remote heart can be supposed to be constant and hence the equation takes the following form:

$$\frac{d}{dr} \left( r \left( \frac{dv}{dr} \right)^n \right) = -\frac{Pr}{\eta_m}.$$

On the integration the equation, we get

$$(4.2) \quad r \left( \frac{dv}{dr} \right)^n = -\frac{Pr^2}{2\eta_m} + A.$$

We know that the velocity of blood flow on the axis of cylindrical arteries is maximum and constant. So that we apply the boundary condition: at  $r = 0$ ,  $v = v_0$  (constant), on equation to get the arbitrary constant  $A = 0$ . Hence the equation takes the following form:<sup>17</sup>

$$(4.3) \quad r \left( \frac{dv}{dr} \right)^n = -\frac{Pr^2}{2\eta_m} \Rightarrow -\frac{dv}{dr} = \left( \frac{Pr}{2\eta_m} \right)^{1/n}.$$

Integrating equation, we get

$$v = -\frac{P}{2\eta_m} \frac{r^{\frac{1}{n}+1}}{(n+1)/n} + B$$

To determine the arbitrary constant B, we apply the no-slip condition on the inner wall of the arteries: at  $r = 0$ ,  $v = 0$ , where  $R$  = radius of vessel, on equation so as to get

$$B = \frac{P^{1/n}}{2\eta_m} \frac{nR_n^{\frac{1}{n}+1}}{(n+1)}$$

Hence the equation takes the following form:

(4.4) 
$$v = \frac{P^{1/n}}{2\eta_m} \frac{n}{(n+1)} (R_n^{\frac{1}{n}+1} - r_n^{\frac{1}{n}+1})$$

This represents the velocity of blood flow in arteries away from the heart, where P is the blood pressure gradient and  $\eta_m$  is the viscosity of blood mixture<sup>18</sup>.

5. Bio-physical Interpretation/ Result and Discussion

The total flow-flux of blood through the transverse section of the arteries is:

(5.1) 
$$\begin{aligned} Q &= \int_0^R v \cdot 2\pi r = \int_0^R \frac{P^{1/n}}{2\eta_m} \frac{n}{(n+1)} (R_n^{\frac{1}{n}+1} - r_n^{\frac{1}{n}+1}) 2\pi r \, dr \\ Q &= \left(\frac{P}{2\eta_m}\right)^{\frac{1}{n}} \frac{n \cdot 2\pi}{n+1} \left(\frac{R_n^{\frac{1}{n}+1}}{2} - \frac{r_n^{\frac{1}{n}+1}}{3n+1}\right) \Bigg|_0^R \\ &= \left(\frac{P}{2\eta_m}\right)^{\frac{1}{n}} \frac{n \cdot 2\pi}{n+1} \frac{(n+1)R_n^{\frac{1}{n}+3}}{2(3n+1)}. \\ Q &= \left(\frac{P}{2\eta_m}\right)^{1/n} \frac{n\pi R_n^{\frac{1}{n}+3}}{(3n+1)}. \end{aligned}$$

6. Patient case history

Patient name- xyz      Age- 27 Year      Sex- Male

Diagnosis: Major  $\beta$  Thalassemia (Cooley’s anemia)

| Date     | Haemoglobin (Hb) | Haematocrit | Blood Pressure | Arteries Pressure Drop in Pascal sec.<br>$\Delta P = p_f - p_i$ | $\Delta P(m\,pa)$ |
|----------|------------------|-------------|----------------|-----------------------------------------------------------------|-------------------|
| 31-05-23 | 6.9              | 20.9        | 109/70         | -2599.779                                                       | 0.0025998         |
| 03-06-23 | 7.6              | 23.2        | 124/80         | -2933.084                                                       | 0.0029331         |
| 05-06-23 | 7.6              | 23.5        | 102/35         | -4466.287                                                       | 0.0044663         |
| 15-06-23 | 8.1              | 24.8        | 98/61          | -2466.457                                                       | 0.0024665         |

Haematocrit ( $H$ ) = 20.9, Viscosity of cell  $\eta_c = 0.01179 \text{ mpa. sec}^{19}$ ,  
Viscosity of plasma  $\eta_p = 0.0192 \text{ mpa. s.}^{20}$

We know that  $\eta_m = \eta_c X + \eta_p(1 - X)$ , where  $X = \frac{H}{100} = 0.209$

$$\eta_m = 0.01179(0.209) + 0.0192(1 - 0.209),$$

$$\eta_m = 0.00246411 + 0.015187$$

$$\eta_m = 0.01765111 \text{ m. pa. s.}$$

Again using this relation and change into the haematocrit

$$\eta_m = \eta_c X + \eta_p(1 - X)$$

$$\eta_m = 0.01179 \left( \frac{H}{100} \right) + 0.0192 \left( 1 - \frac{H}{100} \right)$$

$$\eta_m = 0.0001179H + 0.0192 - 0.000192H$$

$$(6.1) \quad \eta_m = 0.0192 - 0.0000741H$$

From equation (5.1), Put  $P = -\frac{dp}{dz}$ ,

We get

$$Q = \left( -\frac{dP}{2\eta_m dz} \right)^{1/n} \frac{n\pi R^{\frac{1}{n}+3}}{(3n+1)}$$

$$Q = 0.0010269116 \text{ ml/s}^{[20]}$$

$$\text{Length of artery } \Delta Z = 500 \text{ mm}^{[20]}$$

$$\text{Radius of artery } R = 2 \text{ mm}^{[20]}$$

Put the value of  $Q, \Delta P, \Delta Z$  and  $R$  in equation (6.2), we get

$$0.0010269116 = \left( \frac{0.0025998}{2 \times 0.01765111 \times 500} \right)^{\frac{1}{n}} \frac{n\pi(2)^{\frac{1}{n}+3}}{3n+1}$$

$$0.0010269116 = \left( \frac{0.0051996}{17.65111} \right)^{\frac{1}{n}} \frac{n \times 3.14 \times (2)^3}{3n+1}$$

$$0.00004084857 = \frac{n}{3n+1} (0.00029457638)^{\frac{1}{n}}$$

By using Newton Raphson Method<sup>20</sup>

$$(6.2) \quad n = 0.934276$$

$$\begin{aligned} 0.0010269116 &= \left( \frac{\Delta P}{2 \times 0.0192 - 0.0000741H \times 500} \right)^{\frac{1}{n}} \frac{n\pi(2)^{\frac{1}{n}+3}}{3n+1} \\ &= \frac{\Delta P}{2 \times (0.0192 - 0.0000741H) \times 500} \end{aligned}$$

Then

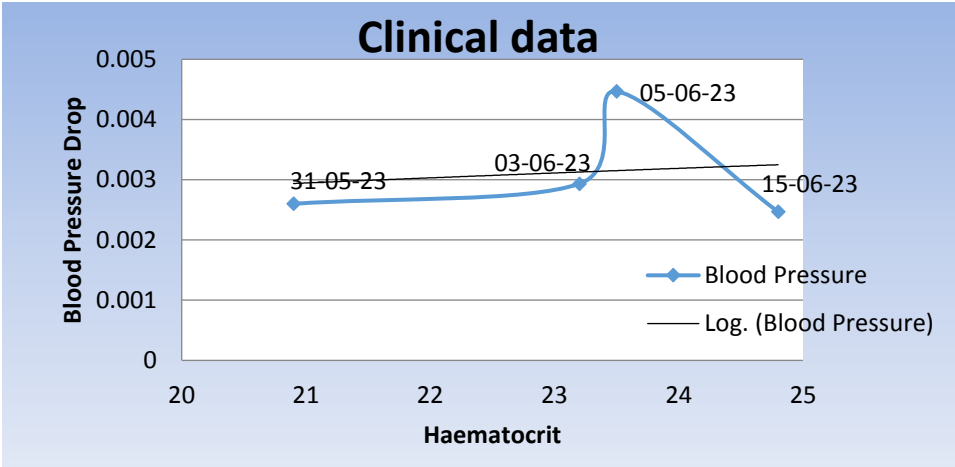
$$0.00011083457 = \frac{\Delta P}{1.92 - 0.00741H}$$

i.e.

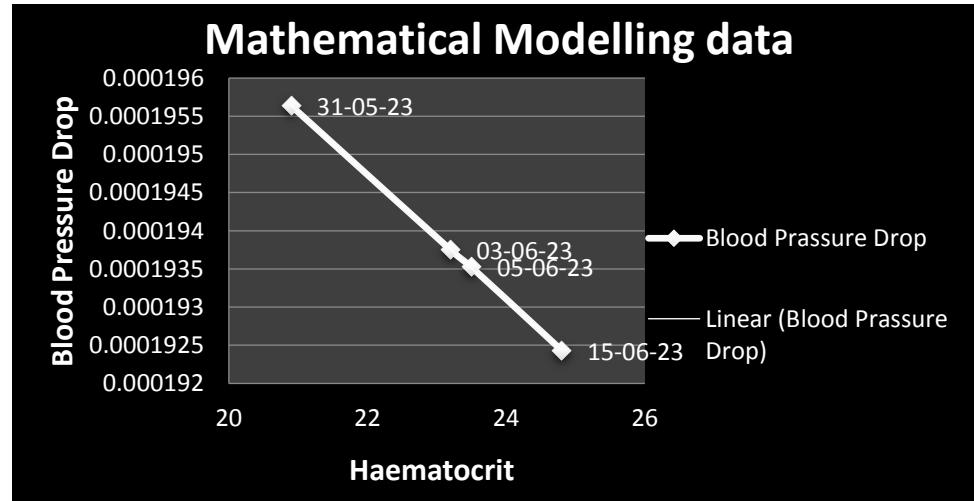
$$\Delta P = 0.00021280237 - 0.00000082128H$$

Table for Hematocrit v/s Pressure drop

| Haematocrit   | 20.9          | 23.2          | 23.5         | 24.8          |
|---------------|---------------|---------------|--------------|---------------|
| Pressure drop | 0.00019563762 | 0.00019374867 | 0.0001935229 | 0.00019243463 |



Graph (a): Relation between Haematocrit v/s Blood Pressure Drop



Graph (b): Relation between Haematocrit v/s Blood Pressure Drop

## 7. Conclusion

This study of haematocrit v/s blood pressure drop in patients with silent thalassemia showing decreasing scenes while data are increasing. Doctor can suggest for heavy dose. The slope of the graph suggests that the dose may increase sharply.

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