

On Decomposition for Curvature Tensor K_{jkh}^i in Generalized $K_{(\mathcal{B})}$ -Quad-Recurrent Space

Adel M. Al-Qashbari

Department of Engineering, Faculty of Engineering and Computer,
University of Science & Technology-Aden, Yemen
Email: Adel_ma71@yahoo.com

Alaa A. Abdallah

Department of Mathematics, Education Faculty, Abyan University, Abyan, Yemen
Email: ala733.ala00@gmail.com

Abdullah S. Saeed

Department of Mathematics, Education Faculty, Aden University, Aden, Yemen
Email: abdullahsaeedmei2022@gmail.com

(Received July 20, 2024, Accepted August 12, 2024)

Abstract: This paper discusses the decomposition for curvature tensor K_{jkh}^i in generalized recurrent space of fourth order. We explore the decomposition tensors $(X^i\Psi_{jkh})$ and $(X_j\Psi_{kh}^i)$ are generalized $\mathcal{B}$ -quad-recurrent tensor under two decomposition forms. If the recurrence covariant tensors α_{lmns} and u_{lmns} have symmetry and skew-symmetric properties, then we get various identities belong to the main space. Moreover, new formula for Cartan's 4th curvature tensor K_{jkh}^i has been obtained under the decomposition.

Keywords: Decomposition of curvature tensor K_{jkh}^i , Generalized $K_{(\mathcal{B})}$ -quad-recurrent space, Symmetry property.

1. Introduction and Preliminaries

The curvature tensor field is a fundamental concept in differential geometry that encapsulates the intrinsic curvature of a manifold. Its decomposition plays a crucial role in understanding the geometric and topological properties of these spaces. In the realm of generalized spaces, the decomposition of the curvature tensor field assumes even greater significance, as it unveils the underlying structure and intricate relationships

within these complex geometries. The generalized recurrence and birecurrence properties of various curvatures tensors have been discussed¹⁻¹³.

Takano¹⁴ investigated the decomposition for the curvatures tensor in recurrent spaces. Over the years, mathematicians have investigated the decomposition of the curvatures tensors in recurrent spaces. Sinha and Singh¹⁵ delved into this concept, specifically exploring and defining the decomposition of the recurrent curvatures tensors field within a Finsler space. Negi and Rawat¹⁶ concentrated their efforts on the decomposition of the recurrent curvatures tensor field in Kählerian spaces. Additionally, some theorems on decomposition of tachibana recurrent space and the decomposition of the curvature tensors in first-order Einstein-Kählerian recurrent spaces studied by authors¹⁷⁻²⁰. Mishra *et al.*²¹ studied the decomposition of curvature tensor field $R_{jkh}^i(x, \dot{x})$ in a Finsler space equipped with non-symmetric connection. Recently, the decomposition of various curvatures tensors in different Finsler spaces has been studied²²⁻²⁶.

Building upon the concept of connection parameters G_{jk}^i , Berwald²⁷ introduced a covariant derivative. This derivative denoted by $\mathcal{B}_k T_j^i$ or $T_{j(k)}^i$, operates on T_j^i is an arbitrary tensor and defined by authors²⁸⁻²⁹.

$$(1.1) \quad T_{j(k)}^i = \partial_k T_j^i - (\partial_r T_j^i) G_k^r + T_j^r G_{rk}^i - T_r^i G_{jk}^r.$$

According to Berwald's theory, the covariant differentiation of the metric function, denoted by F , along any unit vector denoted by ℓ^i and any vector $\dot{x}^i$ simultaneously vanishes.

$$(1.2) \quad \text{a) } F_{(k)} = 0, \quad \text{b) } \ell_{(k)}^i = 0 \quad \text{and} \quad \text{c) } \dot{x}_{(k)}^i = 0.$$

Unlike the standard covariant derivative, Berwald's covariant differentiation of the metric tensor g_{ij} generally does not vanish.

$$(1.3) \quad g_{ij(k)} = -2C_{ijk|h} \dot{x}^h = -2\dot{x}^h C_{ijk(h)}.$$

The tensor C_{ijk} is referred to as the $(h)hv$ -torsion tensor³⁰⁻³¹

$$(1.4) \quad C_{ijk} = \frac{1}{2} \partial_k g_{ij} = \frac{1}{4} \partial_k \partial_i \partial_j F^2.$$

The tensor C_{jk}^i is defined by^{30, 32}

$$(1.5) \quad \text{a) } C_{jsk} = C_{jk}^i g_{is} \quad \text{and} \quad \text{b) } C_{jsk} g^{ji} = C_{sk}^i.$$

Also satisfy the following identities

$$(1.6) \quad \begin{aligned} \text{a) } C_{ijk} \dot{x}^i &= C_{jki} \dot{x}^i = C_{kij} \dot{x}^i = 0, & \text{b) } C_{jk}^i \dot{x}^k &= C_{kj}^i \dot{x}^k = 0, \\ \text{c) } C_{jk}^i \dot{x}_i &= 0, & \text{d) } C_{jr}^r &= C_j \quad \text{and} \quad \text{e) } C_{jk}^i g^{jk} &= C^i. \end{aligned}$$

The vectors $\dot{x}_i$, $\dot{x}^i$ and the metric tensor $g_{ij}(x, \dot{x})$ satisfy

$$(1.7) \quad \begin{aligned} \text{a) } g_{ij} g^{ik} &= \delta_j^k = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}, & \text{b) } \delta_h^i g_{ik} &= g_{hk} . \\ \text{c) } g_{ik} g^{ik} &= \delta_k^k = n, & \text{d) } \delta_h^i g^{hk} &= g^{ik}, & \text{e) } \dot{x}_i \dot{x}^i &= F^2, \\ \text{h) } g_{ij} &= \partial_i \dot{x}_j = \partial_j \dot{x}_i & \text{and} & & \text{i) } \dot{x}_i &= g_{ij}(x, \dot{x}) \dot{x}^j. \end{aligned}$$

The tensor Cartan's 4th curvature tensor, denoted by K_{jkh}^i , is defined by

$$K_{jkh}^i = \partial_h \Gamma_{kj}^{*i} + (\partial_s \Gamma_{jh}^{*i}) G_k^s + \Gamma_{th}^{*i} \Gamma_{kj}^{*t} - \partial_k \Gamma_{hj}^{*i} - (\partial_s \Gamma_{jk}^{*i}) G_h^s - \Gamma_{tk}^{*i} \Gamma_{hj}^{*t} .$$

The curvature tensor K_{jkh}^i possesses the property of skew-symmetry in its last two lower indices, which can be expressed as

$$(1.8) \quad K_{jkh}^i = -K_{jhk}^i .$$

Represented by, the associated curvature tensor K_{ijkh} of tensor K_{jkh}^i is given by the formula

$$(1.9) \quad K_{ijkh} = g_{rj} K_{ikh}^r .$$

The tensor K_{jkh}^i and its associated curvature tensor K_{ijkh} exhibit a set of relationships, as expressed by the following equations

$$(1.10) \quad \begin{aligned} \text{a) } K_{ijkh} + K_{ijhk} &= -2C_{ijs} H_{hk}^s, \\ \text{b) } K_{jikh} + K_{jki h} + K_{jhki} + 2\dot{x}^r (C_{jis} K_{rkh}^s + C_{jks} K_{rih}^s + C_{jhs} K_{rki}^s) &= 0 \\ \text{c) } K_{jkh}^i + K_{hjk}^i + K_{kjh}^i &= 0 \quad \text{and} \quad \text{d) } K_{jikh} + K_{hijk} + K_{kihj} &= 0 . \end{aligned}$$

The K -Ricci tensor K_{jk} , curvature vector K_j and curvature scalar K are interrelated by

$$(1.11) \quad \begin{aligned} \text{a) } K_{jki}^i &= K_{jk}, & \text{b) } K_{jk} \dot{x}^k &= K_j \quad \text{and} & \text{c) } K_{jk} g^{jk} &= K . \end{aligned}$$

The Berwald curvature tensors H_{jkh}^i and torsion tensor H_{kh}^i are satisfy

$$(1.12) \quad \begin{aligned} \text{a) } H_{jkh}^i &= K_{jkh}^i + \dot{x}^m (\partial_j K_{mkh}^i) \quad \text{and} & \text{b) } K_{jkh}^i \dot{x}^j &= H_{kh}^i , \end{aligned}$$

$$(1.13) \quad \begin{aligned} \text{a) } H_{jkh}^i &= \partial_h G_{jk}^i + G_{jk}^r G_{rh}^i + G_{rjh}^i G_k^r - \partial_k G_{jh}^i - G_{jh}^r G_{rk}^i - G_{rjk}^i G_h^r \\ \text{and} \end{aligned}$$

$$\text{b) } H_{kh}^i = \partial_h G_k^i + G_k^r G_{rh}^i - \partial_k G_h^i - G_h^r G_{rk}^i .$$

$$(1.14) \quad \begin{aligned} \text{a) } \partial_r H_{kh}^i &= H_{rkh}^i \quad \text{and} & \text{b) } H_{jkh}^i \dot{x}^j &= H_{kh}^i . \end{aligned}$$

The tensor H_k^i is given by

$$(1.15) \quad \begin{aligned} \text{a) } H_h^i &= 2\partial_h G^i - \partial_s G_k^i \dot{x}^s + 2G_{hs}^i G^s - G_s^i G_h^s \\ \text{and} \end{aligned}$$

$$b) H_h^i = H_{kh}^i \dot{x}^k = -H_{hk}^i \dot{x}^k .$$

The H –Ricci tensor H_{jk} , curvature vector H_k and curvature scalar H are related by

$$(1.16) \ a) H_{jkr}^r = H_{jk} , \quad b) H_{kr}^r = H_k \quad \text{and} \quad c) (n-1)H = H_r^r .$$

In addition to the previously mentioned relationships, the aforementioned tensors also fulfill the following conditions

$$(1.17) \ a) H_{kh(m)}^i + H_{mk(h)}^i + H_{hm(k)}^i = 0 \quad , \quad b) H_{kh}^i = \partial_k H_h^i , \\ c) H_{kh} \dot{x}^k = H_h , \quad d) H_k \dot{x}^k = (n-1)H \quad , \quad f) \dot{x}_i H_j^i = 0 , \\ g) g_{ij} H_k^i = g_{ik} H_j^i \quad \text{and} \quad h) H_{hk} - H_{kh} = H_{ikh}^i .$$

2. The Quad-Recurrence in Finsler Space under Generalized Condition

Riemannian spaces are a special case of Finsler spaces, which provide a distance function that assigns a non-negative real number for all point and non-zero tangent vector in a manifold. Recurrent Finsler spaces are characterized by the existence of a non-zero vector V that satisfies a specific equation involving the Finsler connection.

This paper delves into a specific class of Finsler spaces, Finsler manifolds exhibiting a generalized K -quad recurrent structure, characterized by fourth-order curvature tensors and a non-metric connection. The authors establish formulas for both the 4th-order recurrent curvature tensors. They then proceed to analyze the properties of these newly derived tensors, ultimately achieving some noteworthy results.

We consider a Finsler space denoted by F_n . In this space, Cartan's 4th curvature tensor, denoted by K_{jkh}^i , exhibits a generalized fourth recurrence property with Berwald's connection parameter G_{kh}^i . Mathematically, this property is expressed by the following condition

$$(2.1) \ K_{jkh(l)(m)(n)(s)}^i = a_{lmns} K_{jkh}^i + b_{lmns} (\delta_h^i g_{jk} - \delta_k^i g_{jh}) \\ - 2\dot{x}^r d_{lmn} (\delta_h^i C_{jks} - \delta_k^i C_{jhs})_{(r)} - 2\dot{x}^q \theta_{lms} (\delta_h^i C_{jkn} - \delta_k^i C_{jhn})_{(q)} \\ - 2\dot{x}^q b_{lm} (\delta_h^i C_{jkn} - \delta_k^i C_{jhn})_{(q)(s)} - 2\dot{x}^p \eta_{lms} (\delta_h^i C_{jkm} - \delta_k^i C_{jhm})_{(p)} \\ - 2\dot{x}^p \alpha_{ln} (\delta_h^i C_{jkm} - \delta_k^i C_{jhm})_{(p)(s)} - 2\dot{x}^p \varphi_{ls} (\delta_h^i C_{jkm} - \delta_k^i C_{jhm})_{(p)(n)} \\ - 2\dot{x}^p \mu_l (\delta_h^i C_{jkm} - \delta_k^i C_{jhm})_{(p)(n)(s)} ,$$

where $a_{lmns} = c_{lmn(s)} + c_{lmn} \lambda_s$ and $b_{lmns} = c_{lmn} u_s + d_{lmn(s)}$ are non-zero covariant tensor of fourth-order, $\theta_{lms} = b_{lm(s)}$ and $\eta_{lns} = \alpha_{ln(s)}$ are non-zero covariant tensors of third-order, $\varphi_{ls} = \mu_{l(s)}$ is non-zero covariant tensor of second-order, $(l)(m)(n)(s)$ is Berwald's fourth-order covariant derivative with respect to x^l, x^m, x^n and x^s , respectively.

Definition 2.1. A Finsler space F_n where Cartan's 4th curvature tensor K_{jkh}^i fulfills the condition (2.1), will be referred to as a generalized $K_{(\mathcal{B})}$ -quad-recurrent space. Here, a_{lmns} and b_{lmns} represent non-zero vector fields. Additionally, a tensor satisfying a specific condition will be referred to as a generalized $\mathcal{B}$ -quad-recurrent tensor. For brevity, we will denote such a space and tensor as $GK_{(\mathcal{B})} - QRF_n$ and $G\mathcal{B} - QR$, respectively.

3. Decomposition of Contravariant Vector $X^i(x, \dot{x})$ and Covariant Tensor $\Psi_{jkh}(x, \dot{x})$

In differential geometry, contravariant vectors and covariant tensors are two fundamental concepts used to represent physical quantities and their transformations under coordinate changes. A contravariant vector is a vector that transforms in the opposite direction as a change of coordinates, while a covariant tensor is a tensor that transforms in the same direction as a change of coordinates.

The decomposition of a contravariant vector and a covariant tensor is a technique for expressing them in terms of their components in a particular basis. This decomposition is useful for analyzing the behavior of these quantities under coordinate transformations and for performing calculations in differential geometry.

We consider the decomposition of Cartan's fourth curvature tensor in the form

$$(3.1) \quad K_{jkh}^i = X^i \Psi_{jkh},$$

where the decomposition Ψ_{jkh} satisfies the following

$$(3.2) \quad \Psi_{jkh} = -\Psi_{jhk}.$$

$$(3.3) \quad \Psi_{jkh} x^j = -\Psi_{jhk} x^j = -\Psi_{hk}.$$

$$(3.4) \quad X^i \Psi_{jkh} + X^i \Psi_{khj} + X^i \Psi_{hjk} = 0.$$

$$(3.5) \quad \Psi_{jkh} + \Psi_{khj} + \Psi_{hjk} = 0.$$

Transvecting (3.1) by $\dot{x}^j$, using (3.3), (1.2c) and (1.12b), we get

$$(3.6) \quad H_{kh}^i = X^i \Psi_{kh}.$$

Differentiating (3.1), covariantly with respect to x^l, x^m, x^n and x^s in the sense of Berwald, we get

$$(3.7) \quad K_{jkh(l)(m)(n)(s)}^i = (X^i \Psi_{jkh})_{(l)(m)(n)(s)}.$$

Leveraging the covariant constancy of X^i , we can rewrite above equation as

$$(3.8) \quad K_{jkh(l)(m)(n)(s)}^i = X^i \Psi_{jkh(l)(m)(n)(s)}.$$

In view of equation (2.1), using (3.1) in equation (3.7), we get

$$(3.9) \quad \begin{aligned} (X^i \Psi_{jkh})_{(l)(m)(n)(s)} = & a_{lmns} (X^i \Psi_{jkh}) + b_{lmns} (\delta_h^i g_{jk} - \delta_k^i g_{jh}) \\ & - 2\dot{x}^r d_{lmn} (\delta_h^i C_{jks} - \delta_k^i C_{jhs})_{(r)} - 2\dot{x}^q \theta_{lms} (\delta_h^i C_{jkn} - \delta_k^i C_{jhn})_{(q)} \\ & - 2\dot{x}^q b_{lm} (\delta_h^i C_{jkn} - \delta_k^i C_{jhn})_{(q)(s)} - 2\dot{x}^P \eta_{lns} (\delta_h^i C_{jkm} - \delta_k^i C_{jhm})_{(P)} \\ & - 2\dot{x}^P \alpha_{ln} (\delta_h^i C_{jkm} - \delta_k^i C_{jhm})_{(P)(s)} - 2\dot{x}^P \phi_{ln} (\delta_h^i C_{jkm} - \delta_k^i C_{jhm})_{(P)(n)} \\ & - 2\dot{x}^P u_l (\delta_h^i C_{jkm} - \delta_k^i C_{jhm})_{(P)(n)(s)}. \end{aligned}$$

Therefore, we conclude

Theorem 3.1. *In $GK_{(\mathcal{B})} - QRF_n$, under the decomposition (3.1), if X^i is covariant constant, then the decomposition tensor $(X^i \Psi_{jkh})$ is a generalized $\mathcal{B}$ -quad-recurrent.*

Transvecting (3.9) by x^j , using (1.2c), 3.3), (1.6a) and (1.7i), we get

$$(3.10) \quad (X^i \Psi_{kh})_{(l)(m)(n)(s)} = a_{lmns} X^i \Psi_{kh} + b_{lmns} (\delta_h^i \dot{x}_k - \delta_k^i \dot{x}_h).$$

Since X^i is a covariant constant, we can rewrite the above equation as

$$(3.11) \quad X^i \Psi_{kh(l)(m)(n)(s)} = a_{lmns} X^i \Psi_{kh} + \phi_{lmnshk}^i - \phi_{lmnshk}^i,$$

where $\phi_{lmnshk}^i = b_{lmns} \delta_h^i \dot{x}_k$, $\phi_{lmnshk}^i = b_{lmns} \delta_k^i \dot{x}_h$.

Now, Assuming the tensor ϕ_{lmnshk}^i exhibits a skew-symmetric in the fifth and sixth indices, then (3.11) becomes

$$(3.12) \quad X^i \Psi_{kh(l)(m)(n)(s)} = a_{lmns} X^i \Psi_{kh} + 2 \phi_{lmnshk}^i$$

or

$$(3.13) \quad (X^i \Psi_{kh})_{(l)(m)(n)(s)} = a_{lmns} (X^i \Psi_{kh}) + 2 \phi_{lmnshk}^i.$$

In conclusion, we can determine

Theorem 3.2. *In $GK_{(\mathcal{B})} - QRF_n$, under the decomposition (3.1), and if tensor ϕ_{lmnshk}^i is skew-symmetric, then the decomposition tensor $X^i \Psi_{kh}$ is non vanishing.*

In view of (3.11) and if tensor ϕ_{lmnshk}^i is symmetric, we get

$$(3.14) \quad X^i \Psi_{kh(l)(m)(n)(s)} = a_{lmns} X^i \Psi_{kh}.$$

Above equation can also represent the equation in the following form

$$(3.15) \quad (X^i \Psi_{kh})_{(l)(m)(n)(s)} = a_{lmns} (X^i \Psi_{kh}),$$

Differentiating (1.17a), covariantly with respect to x^m, x^n and x^s in the sense of Berwald, we get

$$(H_{kh}^i)_{(l)(m)(n)(s)} + (H_{hl}^i)_{(k)(m)(n)(s)} + (H_{lk}^i)_{(h)(m)(n)(s)} = 0.$$

By using the equation (3.6) in above equation, can be rewritten as

$$(3.16) \quad (X^i \Psi_{kh})_{(l)(m)(n)(s)} + (X^i \Psi_{hl})_{(k)(m)(n)(s)} + (X^i \Psi_{lk})_{(h)(m)(n)(s)} = 0.$$

Using (3.15) in the above equation we get

$$(3.17) \quad X^i (a_{lmns} \Psi_{kh} + a_{kmns} \Psi_{hl} + a_{hmns} \Psi_{lk}) = 0.$$

Above equation can be written as

$$(3.18) \quad a_{lmns} \Psi_{kh} + a_{kmns} \Psi_{hl} + a_{hmns} \Psi_{lk} = 0, \text{ where } X^i \neq 0.$$

Therefore, we can say

Theorem 3.3. *In $GK_{(\mathcal{B})} - QRF_n$, under the decomposition (3.1) and if X^i is covariant constant, and tensor ϕ_{lmnshk}^i is symmetric, then the decomposition tensor Ψ_{kh} satisfy the identity (3.18).*

Applying cyclic rotation to the indices l, m, k , and h in equation (3.18), yields

$$(3.19) \quad a_{hlms} \Psi_{mk} + a_{mlns} \Psi_{kh} + a_{klms} \Psi_{hm} = 0.$$

$$(3.20) \quad a_{khms} \Psi_{lm} + a_{lhms} \Psi_{mk} + a_{mhks} \Psi_{kl} = 0$$

and

$$(3.21) \quad a_{mkns} \Psi_{hl} + a_{hkns} \Psi_{lm} + a_{lkns} \Psi_{mh} = 0.$$

By using the property of skew-symmetric for the decomposition tensor Ψ_{kh} in its lower indices and adding (3.18), (3.19), (3.20) and (3.21), we get

$$(3.22) \quad (a_{lmns} + a_{mlns}) \Psi_{kh} + (a_{kmns} + a_{mkns}) \Psi_{hl} \\ + (a_{hlms} + a_{lhms}) \Psi_{mk} + (a_{khms} + a_{hkms}) \Psi_{lm} \\ + (a_{hmns} - a_{mhns}) \Psi_{kl} + (a_{klms} - a_{lkms}) \Psi_{mh} = 0.$$

Assuming the recurrence covariant tensor a_{lmns} exhibits symmetry with respect to its first and second indices, the above equation can be rewritten as

$$(3.23) \quad a_{lmns} \Psi_{kh} + a_{mkns} \Psi_{hl} + a_{khms} \Psi_{lm} + a_{hlms} \Psi_{mk} = 0.$$

Thus, we conclude

Theorem 3.4. *In $GK_{(\mathcal{B})} - QRF_n$, under the decomposition (3.1) and assuming X^i is covariant constant and a_{lmns} is symmetric, the identity (3.23), holds for the decomposition tensor Ψ_{kh} .*

Similarly, by cyclically rotating the indices l, m, k , and h in (3.23), we obtain the following equivalent equation

$$(3.24) \quad a_{mnks}\Psi_{hl} + a_{nhks}\Psi_{lm} + a_{hlks}\Psi_{mn} + a_{lmks}\Psi_{nh} = 0.$$

$$(3.25) \quad a_{nkhs}\Psi_{lm} + a_{klhs}\Psi_{mn} + a_{lmhs}\Psi_{nk} + a_{mnhs}\Psi_{kl} = 0.$$

$$(3.26) \quad a_{khls}\Psi_{mn} + a_{hmls}\Psi_{nk} + a_{mnls}\Psi_{kh} + a_{nkls}\Psi_{hm} = 0$$

and

$$(3.27) \quad a_{hlms}\Psi_{nk} + a_{lnms}\Psi_{kh} + a_{nkms}\Psi_{hl} + a_{khms}\Psi_{ln} = 0.$$

Adding (3.23), (3.24), (3.25), (3.26) and (3.27), we get

$$(3.28) \quad [(a_{lmns} + a_{lnms})\Psi_{kh} + (a_{mkns} + a_{mnks})\Psi_{hl} \\ + (a_{nhks} + a_{nkhs})\Psi_{lm} + (a_{klhs} + a_{khls})\Psi_{mn} + (a_{hmls} + a_{hlms})\Psi_{nk}] \\ + [a_{nkms}\Psi_{hl} + a_{khms}\Psi_{ln} + a_{lmhs}\Psi_{nk} + a_{mnls}\Psi_{kh}] \\ + [a_{khns}\Psi_{lm} + a_{hlms}\Psi_{mk} + a_{mnhs}\Psi_{kl}] \\ + [a_{hlks}\Psi_{mn} + a_{lmks}\Psi_{nh} + a_{nkls}\Psi_{hm}] = 0.$$

If the recurrence covariant tensor a_{lmns} is symmetric w. r. t. its first, second and third indices i.e., $a_{lmns} = a_{mns l} = a_{nslm}$, we can express the above equation in the following equivalent form

$$[a_{hlks}\Psi_{mn} + a_{mlks}\Psi_{nh} + a_{nkls}\Psi_{hm}] \\ + [a_{khns}\Psi_{lm} + a_{lhns}\Psi_{mk} + a_{mhns}\Psi_{kl}] \\ + 2[a_{lmns}\Psi_{kh} + a_{mnks}\Psi_{hl} + a_{nkhs}\Psi_{lm} + a_{khls}\Psi_{mn} + a_{hlms}\Psi_{nk}] = 0.$$

By using the equations (3.18) and (3.23) in above equation we get

$$(3.29) \quad a_{lmns}\Psi_{kh} + a_{mnks}\Psi_{hl} + a_{nkhs}\Psi_{lm} + a_{khls}\Psi_{mn} + a_{hlms}\Psi_{nk} = 0.$$

Thus, we conclude

Theorem 3.5. In $GK_{(\mathcal{B})} - QRF_n$, under the decomposition (3.1) and if X^i is covariant constant and under the condition that a_{lmns} is symmetric in its first, second, and third indices, the decomposition tensor Ψ_{kh} satisfies the identity (3.29).

By using the property of skew-symmetric for the decomposition tensor Ψ_{kh} in its lower indices in (3.18), we get

$$(3.30) \quad a_{lmns}\Psi_{kh} + a_{kmns}\Psi_{hl} - a_{hmns}\Psi_{kl} = 0.$$

multiplying above equation by X^i , we get

$$a_{lmns}X^i\Psi_{kh} = a_{hmns}X^i\Psi_{kl} - a_{kmns}X^i\Psi_{hl}.$$

By using the equations (3.6) in above equation, we get

$$a_{lmns}H_{kh}^i = a_{hmns}H_{kl}^i - a_{kmns}H_{hl}^i.$$

By contracting the indices i and l in the above equation and utilizing formula (1.16b), we obtain the following equation

$$a_{imns}H_{kh}^i = a_{hmns}H_k - a_{kmns}H_h.$$

Above equation can be written as

$$(3.31) \quad H_{kh}^i = \alpha_h^i H_k - \alpha_k^i H_h,$$

$$\text{where } \frac{a_{hmns}}{a_{imns}} = \alpha_h^i, \quad \frac{a_{kmns}}{a_{imns}} = \alpha_k^i.$$

Therefore, we conclude

Theorem 3.6. In $GK_{(B)}-QRF_n$, under decomposition (3.1) and if X^i is covariant constant, the $h(v)$ -torsion tensor H_{kh}^i is defined by formula (3.31).

4. Decomposition of Covariant Vector $X_j(x, \dot{x})$ and Mixed Tensor $\Psi_{kh}^i(x, \dot{x})$

The decomposition of a covariant vector and a mixed tensor is a technique for expressing them in terms of their components in a particular basis. This decomposition is useful for analyzing the behavior of these quantities under coordinate transformations and for performing calculations in differential geometry.

We consider the decomposition of Cartan's 4th curvatures tensor in the form

$$(4.1) \quad K_{jkh}^i = X_j \Psi_{kh}^i,$$

where the decomposition tensor Ψ_{kh}^i satisfies the following

$$(4.2) \quad X_j \Psi_{kh}^i + X_k \Psi_{hj}^i + X_h \Psi_{jk}^i = 0,$$

$$(4.3) \quad X_j \Psi_{kh}^i = -X_j \Psi_{hk}^i.$$

From equations (4.2) and (4.3), the decomposition tensor Ψ_{kh}^i satisfies the following

$$(4.4) \quad X_j \Psi_{kh}^i = X_h \Psi_{jk}^i - X_k \Psi_{jh}^i.$$

Transvecting (4.1) by $\dot{x}^j$, using $(X_j \dot{x}^j = X)$, (1.2c) and (1.12b), we get

$$(4.5) \quad H_{kh}^i = X \Psi_{kh}^i.$$

Upon contracting the indices i and h in the above equation (4.1) and employing formula (1.11a), we get the following equation

$$(4.6) \quad K_{jk} = X_j \Psi_{ki}^i.$$

Differentiating (4.1), covariantly w. r. t. x^l, x^m, x^n and x^s in the sense of Berwald, assume that X_j is independent of x^l, x^m, x^n and x^s , we get

$$K_{jkh(l)(m)(n)(s)}^i = X_j \Psi_{kh(l)(m)(n)(s)}^i.$$

In view of equation (2.1), using (4.1) in above equation, we get

$$(4.7) \quad (X_j \Psi_{kh}^i)_{(l)(m)(n)(s)} = \alpha_{lmns} X_j \Psi_{kh}^i + u_{lmns} (\delta_h^i g_{jk} - \delta_k^i g_{jh}) \\ - 2\dot{x}^r u_{lmn} (\delta_h^i C_{jks} - \delta_k^i C_{jhs})_{(r)} - 2\dot{x}^q u_{lms} (\delta_h^i C_{jkn} - \delta_k^i C_{jhn})_{(q)} \\ - 2\dot{x}^p u_{lns} (\delta_h^i C_{jkm} - \delta_k^i C_{jhm})_{(p)} - 2\dot{x}^q u_{lm} (\delta_h^i C_{jkn} - \delta_k^i C_{jhn})_{(q)(s)} \\ - 2\dot{x}^p u_{ln} (\delta_h^i C_{jkm} - \delta_k^i C_{jhm})_{(p)(s)} - 2\dot{x}^p u_{ls} (\delta_h^i C_{jkm} - \delta_k^i C_{jhm})_{(p)(n)} \\ - 2\dot{x}^p \mu_l (\delta_h^i C_{jkm} - \delta_k^i C_{jhm})_{(p)(n)(s)}.$$

Thus, we conclude

Theorem 4.1. *In $GK_{(\mathcal{B})} - QRF_n$, under the decomposition (4.1), the decomposition tensor $X_j \Psi_{kh}^i$ is a generalized $\mathcal{B}$ -quad-recurrent.*

Transvecting (4.7) by $\dot{x}^j$, using $(X_j \dot{x}^j = X)$, (1.2c), (1.6a) and (1.7i), we get

$$(4.8) \quad (X \Psi_{kh}^i)_{(l)(m)(n)(s)} = \alpha_{lmns} X \Psi_{kh}^i + u_{lmns} (\dot{x}_k \delta_h^i - \delta_k^i \dot{x}_h).$$

Contracting the indices i and h in (4.8), using $(\Psi_{ki}^i = \Psi_k)$ and (1.7c), we get

$$(4.9) \quad (X \Psi_k)_{(l)(m)(n)(s)} = \alpha_{lmns} X \Psi_k + u_{lmns} (n - 1) \dot{x}_k.$$

Transvecting by $\dot{x}^k$ (4.9), using $(\Psi_k \dot{x}^k = \Psi)$ and (1.7e), we get

$$(4.10) \quad (X \Psi)_{(l)(m)(n)(s)} = \alpha_{lmns} X \Psi + u_{lmns} (n - 1) F^2.$$

The equations (4.9) and (4.10) show that, the decomposition tensor $(X \Psi_k)$ and decomposition curvature scalar $(X \Psi)$ cannot be simultaneously zero, as this would lead to the contradictory condition of $u_{lmns} = 0$. In conclusion, we get

Theorem 4.2. *In $GK_{(\mathcal{B})} - QRF_n$, under the decomposition (4.1), the decomposition tensors $(X \Psi_k)$ and $(X \Psi)$ are non-vanishing.*

Differentiating (1.17a), covariantly w. r. t. x^m, x^n and x^s in sense of Berwald, we get

$$(4.11) \quad (H_{kh}^i)_{(l)(m)(n)(s)} + (H_{hl}^i)_{(k)(m)(n)(s)} + (H_{lk}^i)_{(h)(m)(n)(s)} = 0.$$

By using the equation (4.5) in the above equation, we get

$$(4.12) \quad (X \Psi_{kh}^i)_{(l)(m)(n)(s)} + (X \Psi_{hl}^i)_{(k)(m)(n)(s)} + (X \Psi_{lk}^i)_{(h)(m)(n)(s)} = 0.$$

Using (4.8) in (4.12), we get

$$(4.13) \quad \alpha_{lmns} X \Psi_{kh}^i + \alpha_{kmns} X \Psi_{hl}^i + \alpha_{hmns} X \Psi_{lk}^i + u_{lmns} (\dot{x}_k \delta_h^i - \delta_k^i \dot{x}_h) \\ + u_{kmns} (\dot{x}_h \delta_l^i - \delta_h^i \dot{x}_l) + u_{hmns} (\dot{x}_l \delta_k^i - \delta_l^i \dot{x}_k) = 0.$$

Considering in equation (4.3), we can rewrite the above equation as

$$\begin{aligned}\alpha_{lmns} X \Psi_{kh}^i &= \alpha_{hmns} X \Psi_{kl}^i - \alpha_{kmns} X \Psi_{hl}^i + u_{lmns} (\delta_k^i \dot{x}_h - \dot{x}_k \delta_h^i) \\ &\quad + u_{kmns} (\delta_h^i \dot{x}_l - \dot{x}_h \delta_l^i) + u_{hmns} (\delta_l^i \dot{x}_k - \dot{x}_l \delta_k^i).\end{aligned}$$

Contracting the indices i and l in the aforementioned equation and applying formula (1.7c), we obtain

$$\begin{aligned}\alpha_{imns} X \Psi_{kh}^i &= \alpha_{hmns} X \Psi_{ki}^i - \alpha_{kmns} X \Psi_{hi}^i + u_{kmns} \dot{x}_h - \dot{x}_k u_{hmns} \\ &\quad + u_{kmns} \dot{x}_h - n \dot{x}_h u_{kmns} + n u_{hmns} \dot{x}_k - \dot{x}_k u_{hmns}.\end{aligned}$$

Multiplying above equation by X_j , we have

$$\begin{aligned}\alpha_{imns} X X_j \Psi_{kh}^i &= \alpha_{hmns} X X_j \Psi_{ki}^i - \alpha_{kmns} X X_j \Psi_{hi}^i \\ &\quad + (n-2) [X_j u_{hmns} \dot{x}_k - X_j \dot{x}_h u_{kmns}].\end{aligned}$$

Using (4.1) and (4.6) in above equation, we have

$$\alpha_{imns} K_{jkh}^i = \alpha_{hmns} K_{jk}^i - \alpha_{kmns} K_{jh}^i + \frac{(n-2)}{X} [X_j u_{hmns} \dot{x}_k - X_j \dot{x}_h u_{kmns}].$$

Above equation can be written as

$$(4.14) \quad K_{jkh}^i = \alpha_h^i K_{jk}^i - \alpha_k^i K_{jh}^i + \frac{(n-2)}{X} [X_j u_h^i \dot{x}_k - X_j \dot{x}_h u_k^i],$$

$$\text{where } \mu_h^i = \frac{u_{hmns}}{\alpha_{imns}}, \quad \mu_k^i = \frac{u_{kmns}}{\alpha_{imns}}, \quad \alpha_h^i = \frac{\alpha_{hmns}}{\alpha_{imns}}, \quad \alpha_k^i = \frac{\alpha_{kmns}}{\alpha_{imns}}.$$

In conclusion, we get

Theorem 4.3. In $GK_{(B)} - QRF_n$, under the decomposition (4.1), the Cartan's 4th curvature tensor K_{jkh}^i defined by formula (4.14).

By cyclically rotating the indices m , k and h in (4.13), we obtain the following equivalent equation, which reflects the transformation induced by the permutation of the indices

$$(4.15) \quad \alpha_{hlms} X \Psi_{mk}^i + \alpha_{mlns} X \Psi_{kh}^i + \alpha_{klms} X \Psi_{hm}^i + u_{hlms} (\dot{x}_m \delta_k^i - \delta_m^i \dot{x}_k) \\ + u_{mlns} (\dot{x}_k \delta_h^i - \delta_k^i \dot{x}_h) + u_{klms} (\dot{x}_h \delta_m^i - \delta_h^i \dot{x}_m) = 0.$$

$$(4.16) \quad \alpha_{khms} X \Psi_{lm}^i + \alpha_{lhms} X \Psi_{mk}^i + \alpha_{mhls} X \Psi_{kl}^i + u_{khms} (\dot{x}_l \delta_m^i - \delta_l^i \dot{x}_m) \\ + u_{lhms} (\dot{x}_m \delta_k^i - \delta_m^i \dot{x}_k) + u_{mhls} (\dot{x}_k \delta_l^i - \delta_k^i \dot{x}_l) = 0$$

and

$$(4.17) \quad \alpha_{mkns} X \Psi_{hl}^i + \alpha_{hkns} X \Psi_{lm}^i + \alpha_{lkns} X \Psi_{mh}^i + u_{mkns} (\dot{x}_h \delta_l^i - \delta_h^i \dot{x}_l) \\ + u_{hkns} (\dot{x}_l \delta_m^i - \delta_l^i \dot{x}_m) + u_{lkns} (\dot{x}_m \delta_h^i - \delta_m^i \dot{x}_h) = 0.$$

By using the property of skew-symmetric for the decomposition tensor Ψ_{kl}^i in its lower indices and adding (4.13), (4.15), (4.16) and (4.17), we get

$$(4.18) \quad (\alpha_{lmns} + \alpha_{mlns})X\Psi_{kh}^i + (\alpha_{kmns} + \alpha_{mkns})X\Psi_{hl}^i \\ + (\alpha_{hlms} + \alpha_{lhms})X\Psi_{mk}^i + (\alpha_{khms} + \alpha_{hkms})X\Psi_{lm}^i \\ + (\alpha_{hmns} - \alpha_{mhns})X\Psi_{kl}^i + (\alpha_{klms} - \alpha_{lkms})X\Psi_{mh}^i = 0.$$

Assuming the recurrence covariant tensors α_{lmns} and u_{lmns} possess symmetry with respect to their first and second indices, we can express the above equation in the following equivalent form

$$(4.19) \quad \alpha_{lmns}X\Psi_{kh}^i + \alpha_{mkns}X\Psi_{hl}^i + \alpha_{khms}X\Psi_{lm}^i + \alpha_{hlms}X\Psi_{mk}^i \\ + u_{lmns}(\dot{x}_k\delta_h^i - \delta_k^i\dot{x}_h) + u_{mkns}(\dot{x}_h\delta_l^i - \delta_h^i\dot{x}_l) \\ + u_{khms}(\dot{x}_l\delta_m^i - \delta_l^i\dot{x}_m) + u_{hlms}(\dot{x}_m\delta_k^i - \delta_m^i\dot{x}_k) = 0.$$

Therefore, we conclude

Theorem 4.4. In $GK_{(B)} - QRF_n$, under the decomposition (4.1), if the recurrence covariant tensors α_{lmns} and u_{lmns} are possess symmetry with respect to their first and second indices, then we have the identity (4.19).

By cyclic rotation of the indices l, m, n, k and h in (4.19), we get

$$(4.20) \quad \alpha_{mnks}X\Psi_{hl}^i + \alpha_{nhks}X\Psi_{lm}^i + \alpha_{hlks}X\Psi_{mn}^i + \alpha_{lmks}X\Psi_{nh}^i \\ + u_{mnks}(\dot{x}_h\delta_l^i - \delta_h^i\dot{x}_l) + u_{nhks}(\dot{x}_l\delta_m^i - \delta_l^i\dot{x}_m) \\ + u_{hlks}(\dot{x}_m\delta_n^i - \delta_m^i\dot{x}_n) + u_{lmks}(\dot{x}_n\delta_h^i - \delta_n^i\dot{x}_h) = 0.$$

$$(4.21) \quad \alpha_{nkhs}X\Psi_{lm}^i + \alpha_{klhs}X\Psi_{mn}^i + \alpha_{lmhs}X\Psi_{nk}^i + \alpha_{mnhs}X\Psi_{kl}^i \\ + u_{nkhs}(\dot{x}_l\delta_m^i - \delta_l^i\dot{x}_m) + u_{klhs}(\dot{x}_m\delta_n^i - \delta_m^i\dot{x}_n) \\ + u_{lmhs}(\dot{x}_n\delta_k^i - \delta_n^i\dot{x}_k) + u_{mnhs}(\dot{x}_k\delta_l^i - \delta_k^i\dot{x}_l) = 0.$$

$$(4.22) \quad \alpha_{khls}X\Psi_{mn}^i + \alpha_{hmls}X\Psi_{nk}^i + \alpha_{mnl s}X\Psi_{kh}^i + \alpha_{nkls}X\Psi_{hm}^i \\ + u_{khls}(\dot{x}_m\delta_h^i - \delta_m^i\dot{x}_h) + u_{hmls}(\dot{x}_n\delta_k^i - \delta_n^i\dot{x}_k) \\ + u_{mnl s}(\dot{x}_k\delta_h^i - \delta_k^i\dot{x}_h) + u_{nkls}(\dot{x}_h\delta_m^i - \delta_h^i\dot{x}_m) = 0$$

and

$$(4.23) \quad \alpha_{hlms}X\Psi_{nk}^i + \alpha_{lmns}X\Psi_{kh}^i + \alpha_{nkms}X\Psi_{hl}^i + \alpha_{ukhms}X\Psi_{ln}^i \\ + u_{hlms}(\dot{x}_n\delta_k^i - \delta_n^i\dot{x}_k) + u_{lmns}(\dot{x}_k\delta_h^i - \delta_k^i\dot{x}_h) \\ + u_{nkms}(\dot{x}_h\delta_l^i - \delta_h^i\dot{x}_l) + u_{ukhms}(\dot{x}_l\delta_n^i - \delta_l^i\dot{x}_n) = 0.$$

By adding (4.19), (4.20), (4.21), (4.22) and (4.23), we get

$$\begin{aligned}
 (4.24) \quad & [(\alpha_{lmns} + \alpha_{lnms})X\Psi_{kh}^i + (\alpha_{mkns} + \alpha_{mnks})X\Psi_{hl}^i \\
 & + (\alpha_{nhks} + \alpha_{nkhs})X\Psi_{lm}^i + (\alpha_{klhs} + \alpha_{khls})X\Psi_{mn}^i \\
 & + (\alpha_{hmls} + \alpha_{hlms})X\Psi_{nk}^i + (u_{lmns} + u_{lnms})(\dot{x}_k\delta_h^i - \delta_k^i\dot{x}_h) \\
 & + (u_{mkns} + u_{mnks})(\dot{x}_h\delta_l^i - \delta_h^i\dot{x}_l) + (u_{nhks} + u_{nkhs})(\dot{x}_l\delta_m^i - \delta_l^i\dot{x}_m) \\
 & + (u_{klhs} + u_{khls})(\dot{x}_m\delta_n^i - \delta_m^i\dot{x}_n) + (u_{hmls} + u_{hlms})(\dot{x}_n\delta_k^i - \delta_n^i\dot{x}_k)] \\
 & + [\alpha_{nkms}X\Psi_{hl}^i + \alpha_{khms}X\Psi_{ln}^i + \alpha_{lmhs}X\Psi_{nk}^i + \alpha_{mnls}X\Psi_{kh}^i \\
 & + u_{nkms}(\dot{x}_h\delta_l^i - \delta_h^i\dot{x}_l) + u_{khms}(\dot{x}_l\delta_n^i - \delta_l^i\dot{x}_n) \\
 & + u_{lmhs}(\dot{x}_n\delta_k^i - \delta_n^i\dot{x}_k) + u_{mnls}(\dot{x}_k\delta_h^i - \delta_k^i\dot{x}_h)] \\
 & + [\alpha_{khns}X\Psi_{lm}^i + \alpha_{hlms}X\Psi_{mk}^i + \alpha_{mnhs}X\Psi_{kl}^i + u_{khns}(\dot{x}_l\delta_m^i - \delta_l^i\dot{x}_m) \\
 & + u_{hlms}(\dot{x}_m\delta_k^i - \delta_m^i\dot{x}_k) + u_{mnhs}(\dot{x}_k\delta_l^i - \delta_k^i\dot{x}_l)] \\
 & + [\alpha_{hlks}XT_{mn}^i + \alpha_{lmks}X\Psi_{nh}^i + \alpha_{nkls}X\Psi_{hm}^i + u_{hlks}(\dot{x}_m\delta_n^i - \delta_m^i\dot{x}_n) \\
 & + u_{lmks}(\dot{x}_n\delta_h^i - \delta_n^i\dot{x}_h) + u_{nkls}(\dot{x}_h\delta_m^i - \delta_h^i\dot{x}_m)] = 0.
 \end{aligned}$$

If the recurrence covariant tensors α_{lmns} and u_{lmns} are symmetric with respect to their first, second and third indicators, then the equation above can be equivalently expressed as

$$\begin{aligned}
 (4.25) \quad & \alpha_{khns}X\Psi_{lm}^i + \alpha_{lhns}X\Psi_{mk}^i + \alpha_{mhns}X\Psi_{kl}^i + u_{khns}(\dot{x}_l\delta_m^i - \delta_l^i\dot{x}_m) \\
 & + u_{lhns}(\dot{x}_m\delta_k^i - \delta_m^i\dot{x}_k) + u_{mhns}(\dot{x}_k\delta_l^i - \delta_k^i\dot{x}_l) + \alpha_{hlks}X\Psi_{mn}^i \\
 & + \alpha_{mlks}X\Psi_{nh}^i + \alpha_{nkls}X\Psi_{hm}^i + u_{hlks}(\dot{x}_m\delta_n^i - \delta_m^i\dot{x}_n) \\
 & + u_{mlks}(\dot{x}_n\delta_h^i - \delta_n^i\dot{x}_h) + u_{nkls}(\dot{x}_h\delta_m^i - \delta_h^i\dot{x}_m) + \alpha_{nkms}X\Psi_{hl}^i \\
 & + \alpha_{khms}X\Psi_{ln}^i + \alpha_{hlms}X\Psi_{nk}^i + \alpha_{lnms}X\Psi_{kh}^i \\
 & + u_{nkms}(\dot{x}_h\delta_l^i - \delta_h^i\dot{x}_l) + u_{khms}(\dot{x}_l\delta_n^i - \delta_l^i\dot{x}_n) + u_{hlms}(\dot{x}_n\delta_k^i - \delta_n^i\dot{x}_k) \\
 & + u_{lnms}(\dot{x}_k\delta_h^i - \delta_k^i\dot{x}_h) + 2[\alpha_{lmns}X\Psi_{kh}^i + \alpha_{mnks}X\Psi_{hl}^i + \alpha_{nkhs}X\Psi_{lm}^i \\
 & + \alpha_{khls}X\Psi_{mn}^i + \alpha_{hlms}X\Psi_{nk}^i + u_{lmns}(\dot{x}_k\delta_h^i - \delta_k^i\dot{x}_h) \\
 & + u_{mnks}(\dot{x}_h\delta_l^i - \delta_h^i\dot{x}_l) + u_{nkhs}(\dot{x}_l\delta_m^i - \delta_l^i\dot{x}_m) \\
 & + u_{khls}(\dot{x}_m\delta_n^i - \delta_m^i\dot{x}_n) + u_{hlms}(\dot{x}_n\delta_k^i - \delta_n^i\dot{x}_k)] = 0.
 \end{aligned}$$

Using equations (4.13) and (4.19) in above equation, we get

$$\begin{aligned}
 (4.26) \quad & \alpha_{lmns}X\Psi_{kh}^i + \alpha_{mnks}X\Psi_{hl}^i + \alpha_{nkhls}X\Psi_{lm}^i + \alpha_{khlsl}X\Psi_{mn}^i \\
 & + \alpha_{hlms}X\Psi_{nk}^i + u_{lmns}(\dot{x}_k\delta_h^i - \delta_k^i\dot{x}_h) + u_{mnks}(\dot{x}_h\delta_l^i - \delta_h^i\dot{x}_l) \\
 & + u_{nkhls}(\dot{x}_l\delta_m^i - \delta_l^i\dot{x}_m) = 0.
 \end{aligned}$$

In conclusion, we get

Theorem 4.5. In $GK_{(\mathcal{B})}-QRF_n$, under the decomposition (4.1), if the recurrence covariant tensors α_{lmns} and u_{lmns} are symmetric in first, second and third indicators, the decomposition tensor $X\Psi_{kh}^i$ satisfies the identity (4.26).

5. Conclusion

The decomposition tensors $(X^i\Psi_{jkh})$ and $(X_j\Psi_{kh}^i)$ are a generalized $\mathcal{B}$ -quad-recurrent tensor under two decomposition forms have been obtained. Also, we prove the decomposition for different tensors are non-vanishing in $GK_{(\mathcal{B})} - QRF_n$.

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